SOME ASPECTS OF MINKOWSKI DISTANCES

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ABSTRACT

This paper discusses implications of the Minkowski model for distances. It was written when the author wanted to solve some problems for himself. Other literature about the subject was not consulted. This explains why the paper does not contain a list of references.

The paper starts with an exposition of the Minkowski model (Sections 1 and 2).
Section 3 contains some thoughts about how Minkowski distances based upon a different choice of the Minkowski parameter n are related when the model is applied to the same set of data.

Sections 4 to 6 deal with more specific problems. Section 4 is about the problem: given two points A and B, what characterizes points Z such that these points have the same Minkowski distance to A and B?

Section 5 discusses the following problem. Let O be the point with coordinates (0, 0), whereas Z is a point with coordinates $Z = (x, y)$. Let d be the Minkowski distance between O and Z, for a given choice of parameter n. Let x have some fixed value. then: how is d related to y?

Section 6 is about additivity of distances. Given two points A and B, what characterizes points Z when it is required that the Minkowski distances between Z and A and Z and B, add up to the Minkowski distance between A and B? Or: how does it look like when it is required for distances that $\underline{AZ} + \underline{ZB} = \underline{AB}$?

Section 7, finally, is about the interpretation of Minkowski models when they are applied to responses of subjects who are asked to assess the dissimilarity (or similarity) between pairs of objects. The argument in Section 7 is that even when a Minkowski model gives a good *description* of the empirical data, it does not follow how those data can be *explained*. In fact, one should try to find out which "intuitive rules" are used by the subject, and how it may happen that those intuitive rules can result into responses that match with a description based upon the Minkowski model.

1. Definition

Let (x_i, y_i) and (x_j, y_j) be two points in a two-dimensional space. The general *Minkowski distance* d_{ij} between these two points is defined as

$$d_{ij} = \{|x_i - x_j|^n + |y_i - y_j|^n\}^{1/n}.$$

For $n = 2$ this expression defines classical Euclidean distance. For given fixed values (x_i, y_i) and a fixed value of d_{ij}, it follows that point (x_j, y_j) will be located on a circle, with (x_i, y_i) as its center, and with radius d_{ij}. See Figure 1A. In this figure, point (x_i, y_i) is taken as the point (0, 0), the origin of the space – without loss of generality because (x_i, y_i) always can be selected as the origin.

This routine will be maintained in the sequel, in particular in Section 2. It has the advantage that it simplifies notation, in the sense that the generalized distance between (0, 0) and (x, y) can be written as $d = (| x |^n + | y |^n)^{1/n}$.

2. Illustrations of various choices of n

2.1 $n > 0$

Figure 1 shows how the locus of points (x, y) at fixed Minkowski distance from (0, 0) looks like, for various choices of the Minkowski parameter n. As said above, for $n = 2$ this locus becomes a circle, as shown in Figure 1A. With $n = 1$ it becomes a rhombus (oblique square), as shown in Figure 1B. The Minkowski distance with $n - 1$ is often called a 'city block' distance (because it corresponds to the distance one has to walk from one place to another, in a city where the streets form a rectangular grid). With $1 < n < 2$ the locus becomes a figure somewhere between the circle and the rhombus. This is illustrated in Figure 1C for $n = 1.5$.

With $n > 2$ the locus becomes a figure somewhere between a circle and an upright square, as shown in Figure 1D for $n = 3$. In fact, when $n \to \infty$, the locus becomes an upright square, as shown in Figure 1E. This choice of n implies that only the largest value of $| x |$ or $| y |$ is taken into account; the smallest is ignored.

The choice $0 < n < 1$ implies that the figure will look like a "star", as illustrated in Figure 1F for $n = .5$. To the extent that n becomes closer to 0, the star will become more narrow. It will look almost like a "cross" when n is very close to 0.

The choice $n = 0$ is a rather special one. It will be discussed later, in Section 2.3.

2.2 $n < 0$

In principle one might also take a negative value of n. In the following this shall be written as $n = -m$, where $m > 0$. The equation for a Minkowski distance then becomes:

$$d = (| x |^{-m} + | y |^{-m})^{-1/m}$$

or

$$d^{-m} = | x |^{-m} + | y |^{-m}$$

or

$$(1/d)^{m} = | 1/x |^{m} + | 1/y |^{m}.$$

The latter expression shows that the "ordinary" Minkowski definition (with parameter $n > 0$) now applies to the *inverses* of d, x, and y. Clearly, the values of d, $| x |$, and $| y |$ can be interpreted as measures of *dissimilarity*. Their inverses therefore can be looked upon as measures of *similarity*.

Figure 1G shows the equi-distance curve for $n = -m = -1$. For a given value of d, the curve again looks like a star, but now a star of which the four branches are outside the asymptotes $x = \pm d$ and $y = \pm d$. Figure 1H shows the limiting case where $n \to -\infty$. The figure also shows that d now depends only upon the smallest value of $| x |$, or $| y |$, and the largest value of these two is ignored.

2.3 $\underline{n = 0}$

The choice $n = 0$ produces a degenerate case. Suppose that $n \to +0$, and that x and y are both positive. It then can be shown that

$$x^n \to 1 + \alpha$$

where α is a very small positive number. Similarly

$$y^n \to 1 + \beta$$

where ß is a very small positive number. It then follows that

$$d = (x^n + y^n)^{1/n} \to (2 + \alpha + \beta)^{\infty} = \infty.$$

However, when $n \to -0$, we obtain

$$x^n \to 1 - \alpha$$
$$y^n \to 1 - \beta$$

$$d \to (2 - \alpha - \beta)^{-\infty} \to 0.$$

We thus see that the infinitely small step from $n \to +0$ to $n \to -0$ has a dramatic effect upon the value of d: it changes from ∞ to 0. The choice $n \to +0$ therefore seems to imply that each two objects are infinitely dissimilar, and that there is no similarity between two different objects. Whereas the choice $n \to -0$ implies that there is no difference between any two objects.

We shall come back to this somewhat paradoxical finding in Section 7.

3. Relation between distances based upon two different values of the Minkowski parameter

3.1 Boundaries

Suppose in a plane a number of points are given. Distances d_m between pairs of points can be calculated on the basis of some selected Minkowski parameter m. But it is also possible to calculate distances d_n, based upon the choice of parameter $n \neq m$. What are the relations between values of d_m and those of d_n?

Distances can be plotted in a graph with d_m as the abscissa and d_n as the ordinate. It is trivial to remark that points in such a graph will be located on a straight line with slope 1 when $n = m$. Points may deviate from that line $d_m = d_n$ to the extent that n becomes more different from m.

Assume that $n > m > 0$. Also, let x be the horizontal distance between two points in the original plot, whereas y is the vertical distance. It then follows that $d_n = d_m$ if either $x = 0$ or $y = 0$. Such cases will appear in the plot of d_n versus d_m as points on the line $d_n = d_m$, with slope 1. Other points will be located below this line.

To illustrate the latter, take the situation where $m = 1$, and $n = 2$. Then d_1 is defined as $(x + y)$: the sum of two rectangular sides of a right-angled triangle. However, d_2 is given by the length of the hypothenuse of that triangle, and it thus follows that $d_2 < d_1$.

More in general, it can be shown that the ratio d_n/d_m is bounded by a line with slope

$$s = 2^{(m - n)/mn}$$

In the plot of d_n with respect to d_m, this line with slope s forms a *lower* bound, whereas the line with slope 1 is the *upper* bound. Points on the line with slope s will be found when $x = y$. This is not only true when $n > m > 0$, but also when $0 > n > m$.

On the other hand, suppose that $m > 0$, and $n < 0$. In the plot of d_n versus d_m, the line with slope s now becomes an upper bound (for those cases where $x = y$), whereas the lower bound is given by the line $d_n = 0$ (for those cases where x and/or y is equal to zero).

As an illustration, take the 10 points (v, w) of which the coordinates are given in the following table:

v:	3	4	5	5	5	6	6	6	7	7
w:	4	5	6	7	3	5	7	4	6	3

In this example, the correlation between v and w is negligibly small ($r = .059$). For the 45 pairs of points, distances d_n (with $n = 2$) and d_m (with $m = 1$) can be calculated. The correlation between d_n and d_m is found to be quite large, equal to .953.

A graph of the 45 pairs is shown in Figure 2. The numbers in this graph indicate how many pairs are located on that position. Clearly, the plot has the shape of a cone, bounded by a line with slope 1, and another line with slope $s = 2^{-m/n} = .707$. On this latter boundary, pairs are located where the absolute difference in v is equal to that in w. For instance, there are 8 pairs where these absolute differences are both equal to 1, so that $d_m = 2$, and $d_n = 2^{1/2} = 1.414$. On the boundary with slope 1 we find pairs where one of the differences is equal to zero. For instance, there are 3 pairs where one of the two differences is equal to zero, whereas the other differences has absolute value equal to 1. For these 3 pairs it follows that $d_n = d_m = 1$. In fact, the graph shows that there are 8 + 2 + 1 = 11 pairs where the differences are equal, whereas for 3 + 4 + 4 + 1 = 12 pairs one of the two differences is equal to 0.

The main point of this illustration is that d_n and d_m are highly correlated, even when there is only a very small correlation between the original coordinates v and w.

3.2 Rank order relations between d_n and d_m

The argument in the previous Section 3.1 implies that there will be a positive correlation between d_n and d_m. It also follows that d_n and d_m must have about the same rank order. Nevertheless it may happen that two distances based on n may have the reversed order when these distances are based on m.

In order to illustrate such reversals, we take the example where $m = 1$ and $n = 2$. The illustration is based upon data given in Table 1. This example refers to 5 points in a plane, labeled A to E, and with coordinates given in Table 1A. City block distances are given in Table 1B, together with their rank order. Euclidean distances are given in Table 1C also with their rank order.

Figure 3 shows a plot of the 5 original points, based upon Table 1A. Clearly, the city-block distance between points C and E is the same as the Euclidean distance - the reason being that these two points differ in only one dimension. On the other hand, when the line connecting two points is very much oblique, then the Euclidean distance between them will be relatively much smaller than their city-block distance.

More generally, take two points V and W. Let their city-block distances from the origin O be denoted as $(OV)_1$ and $(OW)_1$, whereas their Euclidean distances from O are given by $(OV)_2$ and $(OW)_2$. The question is: when will it happen that the order of $(OV)_1$ and $(OW)_2$ is the reversal of the order of $(OV)_2$ and $(OW)_2$?

The answer is given in Figure 4. This figure shows a point A which has the same city-block distance from O as other points located on the line drawn through A, and with slope -1. On the other hand, points located on the circle through A (and with O as center) have the same Euclidean distance from O.

Take a point V located outside the circle, but closer to O than points on the line with slope -1. Compare with a point W located inside the circle, but outside the straight line. Clearly:

$$(OW)_1 > (OV)_1$$

whereas

$$(OW)_2 < (OV)_2$$

which shows that there is order reversal.

The data in Table 1 give an illustration when distances <u>EC</u> and <u>ED</u> are compared. In Euclidean space <u>EC</u> is larger than <u>ED</u>. But in city-block space <u>EC</u> is smaller than <u>ED</u>. Or, if a circle is drawn with E as its center and with radius 5.9, then point D is located *inside* that circle, whereas D is located *above* the line with slope 1 drawn through C.

3.3 Solutions for coordinates of points if the order of their distances is given

Suppose there are k objects, and the rank order of the $k(k-1)/2$ distances between pairs of points is known. It is a well-known problem in "non-metric scaling" whether it is possible to plot the k points in a space with g dimensions, in such a way that the distances in this plot have the same rank order as the one initially given.

It can be shown that such a plot is always possible if we take $g = (k-1)$. However, such a solution is not very interesting. It will be much more attractive to find a solution where g is much smaller than $(k-1)$. For instance, if the rank order of the 190 dissimilarities between $k = 20$ objects is known, will it be possible to plot 20 points in a space with 2 dimensions in such a way that the 190 distances between these points have the same rank order as the 190 dissimilarities?

It often happens that such a solution (with relative small value of g) is not perfect: the two rank orders then are very similar but not completely identical. The g-dimensional solution then is said to show some amount of "stress". (We shall not discuss the question of how such a stress should be measured.)

Suppose that a Minkowski solution with parameter $n = 2.12$ shows no stress at all, whereas a solution with $n = 2$ does show some stress. Does this imply that the solution with $n = 2.12$ is "better" than that with $n = 2$? In a strictly formal sense the answer to this question must be: "yes". However, it also may happen that the observed rank order of the dissimilarities is subject to some amount of observational error. This might imply that the solution with $n = 2$ is ultimately the correct one, although it happens that on the basis of the fallible data a solution with $n = 2.12$ shows less stress. The latter solution thus shows a "gain" compared with the solution $n = 2$. But it could be true that this gain is fallacious, in the sense that it is no more than a "capitalization of error".

Obviously, researchers who want to present results on the basis of a spatial map with g dimensions, have an outspoken preference for a solution in which g is integer. They will therefore prefer a solution with $g = 2$ and some stress, above a solution with $g = 2.12$ without stress. It is, after all, rather difficult to imagine how a space with 2.12 must look like.

On the other hand, one might perhaps argue that the Minkowski model, when it fits, is no more than an algebraic description of the data. It then is not necessary that this description should require a more of less easy "visual" interpretation in terms of spatial dimensions. It remains possible that a subject, when asked to assess (dis)similarities, is guided by rules which are incompatible with some simple spatial representation.

Section 7 of this paper comes back to these issues.

4. Points midway between two given points

4.1 $n > 1$

Let A and B be two points in a two-dimensional space. Without loss of generality take $A = (v, -w)$ and $B = (-v, w)$. In the examples below we shall take $v = 1$, and $w = 3$.

A point $C = (x, y)$ is located midway between A and B if the distances $\underline{AC}$ and $\underline{BC}$ are equal. In Euclidean space, where $n = 2$, the well-known solution is that such points C must be located on a straight line, orthogonal to the line that connects A and B, and passing through point (0, 0). For the example this is the line $y = (v/w)x = (1/3)x$. Figure 5A shows this (trivial) solution.

Figure 5B shows the solution for $n = 1$. The line for points C now becomes a broken line. It has slope 1 within the rectangle of which A and B are corners; it is a horizontal line outside this rectangle. It may be noted that this broken line passes through the points (w, v) and $(-w, -v)$. In fact, it is easy to show that the latter is true for any value of n.

Figure 5C gives the solution for $n \to \infty$. Again it is a broken line. It conforms to $y = 0$ as long as $(v - w) \leq x \leq (w - v)$. For $x > (w - v)$ or $x < (v - w)$ it becomes a line with slope 1. That this solution is valid, may become easier to understand by remembering that with $n \to \infty$, the distance $\underline{AC}$ depends only upon the vertical distance between these two points when this vertical distance is longer than the horizontal distance, but is determined by the horizontal distance when the latter is larger than the vertical one.

With $2 < n < \infty$ the solution for points C at equal distance from A and B becomes a curved line, as shown in Figure 5D for $n = 3$. Obviously, the curve is bounded by the solutions for $n = 2$ and $n = \infty$.

When $1 < n < 2$ the solution is also a curved line, now bounded by the solutions for $n = 1$ and $n = 2$. Figure 5E gives the illustration for $n = 1.5$.

4.2 $\underline{0 < n < 1}$

With $0 < n < 1$, the curve for points C midway between A and B falls apart into three separate branches. The reason why this happens, is explained in Figure 6, where A = (1, -3), B = (-1, 3), and n = .5.

Figure 6A shows the two stars around A and B, where $d = 3 + 5^{1/2} = 5.24$. It is clear that for smaller values of d the two stars will not intersect. In other words, points such that $\underline{AC} = \underline{CB} < 5.24$ do not exist. The figure shows the solution for d where a corner of one star just touches the side of the other star.

Figure 6B shows the two stars when $d = (1^{1/2} + 3^{1/2})^2 = 7.64$. The two stars are tangential to each other at the point (0, 0). But there are also two intersections, one where $x < 1$, and the other one where $x > 1$.

If d is further increased, Figure 6C is obtained for $d = 8$. At points (1, 5) and (-1, -5) the corner of one star again touches a side of the other star. But there are also two other points of intersection, one where $x > 1$ and $0 < y < 5.24$, the other one where $x < -1$ and $0 > y > -5.24$.

Clearly, if d is still further increased, six points of intersection will be found. Two of them will have value $y > 5$, and either $x > 1$ or $0 < x < 1$. If $C = (x, y)$ is such an intersection, then point $C = (-x, -y)$ also will be an intersection.

Comparison between the three figures 6A to 6C shows that no solutions for C can be found when $3.24 < y < 5$, or when $-3.24 > y > -5$.

Figure 7 summarizes the results. It shows a "middle branch" which passes through (0, 0) and has turning points at (1, 2.24) and (-1, -2.24). These two points correspond to the points of touch shown in Figure 6A. Figure 7 also shows an "upper branch", with a turning point at (1, 5) - as illustrated in Figure 6C. The latter figure also shows why the "lower branch" in Figure 7 has a turning point at (-1, -5). Figure 6B shows why the middle branch of Figure 7 passes through point (0, 0).

It may be mentioned that the turning points in Figure 7 become closer to A (or B), to the extent that $n \rightarrow 0$.

Two other remarks may be made.

(a) Why do we not find an upper (or lower) branch when $n > 1$? Why do we find only a "middle branch" in the Figures 5A to 5E?

The formal answer to this question is that there is also an upper branch when $n > 1$, but its "turning point" is at $(1, \infty)$. We then have $\underline{AC} = \underline{CB} = \infty$. The same formal answer is valid for a lower branch, with "turning point" at $(-1, -\infty)$.

(b) Clearly, in Figure 7 we have branches with asymptotes at $x = 0$ and $y = 0$. They are valid when $x \to \pm\infty$ so that $y \to \pm 0$, or when $x \to \pm 0$ and $y \to \pm\infty$. But what is the asymptote when $x \to \pm\infty$, and also $y \to \pm\infty$?

In Figure 7 this is the asymptote $y = 9x$, an asymptote for both the upper and the lower branch. More in general, it can be shown that it is the asymptote $y = sx$, where

$$s = (w/v)^{1/(1-n)}$$

so that in the numerical example of Figure 7, where $v = 1$, $w = 3$, and $n = .5$, we obtain $s = 3^2 = 9$.

The solution for s above can be proved on the basis of the general approximative equality:

$$(p + q)^{1/m} \cong p^{1/m} + q/mp^{1-1/m}$$

given that $m > 1$, and that $p >> q$.

Applied to the situation of Figure 7, the equality $\underline{AC} = \underline{BC}$ for a point $C = (x, y)$, and given values of v and w where $A = (v, -w)$ and $B = (-v, w)$, implies:

$$(x + v)^n + (y - w)^n = (x - v)^n + (y + w)^n$$

where $0 < n < 1$ is the Minkowski parameter. Now write $m = 1/n$, so that $m > 1$. Also, let $x \to \infty$, and $y \to \infty$. The equality above then can be approximated as

$$v/mx^{1-1/m} - w/my^{1-1/m} = v/mx^{1-1/m} + w/my^{1-1/m}$$

with implies that

$$y/x = (w/v)^{m/(m-1)}$$

or

$$y/x = (w/v)^{1/(1-n)}$$

and this is the equation for s shown above.

This derivation is also valid for $x \to -\infty$ and $y \to -\infty$.

4.3 $\underline{n = 0}$

The problem we encounter here is that $n \to +0$ tends to make the distance between two points close to ∞. For example, the distance between (0, 0) and $Z = (x, y)$ is given by

$$d = (x^n + y^n)^{1/n}$$

and when $n = +0$, we obtain $d = (x^0 + y^0)^\infty = (1 + 1)^\infty = \infty$.

However, when $n \to 0$, we shall find that $x^n = (1 + \alpha)$, where α is a very small number (positive when $x > 1$, and negative when $x < 1$).

Again, take the example where $A = (v, -w)$ and $B = (-v, w)$. Clearly, when $0 < x < v$ and $y > w$, the distances between a point $C = (x, y)$ and the two points A and B are equal when

$$(x + v)^n + (y + w)^n = (v - x)^n + (y - w)^n$$

so that for $n \to 0$, we may write

$$(1 + \alpha) + (1 + ß) = (1 + \psi) + (1 + \phi)$$

where α, ß, ψ and ϕ are (positive or negative) values close to 0.

It follows that

$$1 + (\alpha + ß)/2 = 1 + (\psi + \phi)/2.$$

However, given that $(x + v)^n = 1 + \alpha$, it must be true that

$$\{(x + v)^n\}^{1/2} = 1 + \alpha/2$$

since $(1 + \alpha/2)^2 = 1 + 2 + \alpha^2/4$, where α^2 is negligibly small. In the same way

$$(x + v)^n(y + w)^n = (1 + \alpha)(1 + ß) = 1 + \alpha + ß$$

because $\alpha ß$ is negligibly small. Therefore we have the result that

$$1 + (\alpha + ß)/2 = \{(x + v)^n(y + w)^n\}^{1/2}$$
$$1 + (\psi + \phi)/2 = \{(v - x)^n(w - y)^n\}^{1/2}.$$

It thus follows that

$$(x + v)(y + w) = (v + x)(w - y)$$

or

$$xy = vw.$$

The conclusion is that points C located on the hyperbola $xy = vw$ have equal distance from A and B when $n \to +0$. It is easy to show that the derivation of this result, given above, is not only valid when $0 < x < v$ and $y > w$, but also for the other parts of the curve. The resulting hyperbola has asymptotes at $x = 0$ and $y = 0$. One of its branches passes through the point (v, w), and the other one through $(-v, -w)$. Moreover, when $C = (x, y)$ is located on one branch, then the point $(-x, -y)$ will be located on the other branch.

In addition, take points $C = (x, y)$ located on the straight line $y = (w/v)x$.

It is not difficult to prove that such points also satisfy the requirement that $\underline{AC} = \underline{CB}$. The general solution for $n \to +0$ is shown in Figure 8. The figure contains the hyperbola $xy = vw$, and also the straight line with slope (w/v).

Remember that $n \to +0$ implies that distances between two points tend to become very large. In order to illustrate what is happening, take the example above (with $v = 1$, $w = 3$), and let $n = .1$. A point $C = (3, 9)$ is located on the straight line with slope $w/v = 3$. We then obtain distances $\underline{AC} = 5221.7$ and $\underline{BC} = 5026.9$. A small increase of the second coordinate, so that $C = (3, 10.0889)$ has the effect that the distances become equal: $\underline{AC} = \underline{CB} = 5475.2$. Note that all such distances are already very large.

Compare this result with $n \to -0$, such as $n = -.1$. If we take $C = (3, 9)$ again, then $\underline{AC} =$ 4.596E-3, and $\underline{BC} =$ 4.774E-3. A shift towards $C = (3, 8.2160)$ gives the solution where $\underline{AC} = \underline{BC} =$ 4.457E-3. All such distances now become extremely small.

Also, for $x > v$ and $y > w$, the solution for points C where $\underline{AC} = \underline{BC}$ has asymtote with slope $(w/v)^{1/(1-n)}$. When $n = 0$, the curve coincides with this asymptote, and becomes $y = 3x$. With $n = .1$ the asymptote has slope $3^{1/.9} = 3.3895$, somewhat steeper than the line $y = (w/v)x = 3x$. A point on the asymptote is (3, 10.1685), which shows that the correct solution $C = (3, 10.0889)$ is located slightly *below* the asymptote. With $n = -.1$, the asymptote has slope $3^{1/1.1} = 2.7149$, somewhat flatter than 3. A point on the asymptote is (3, 8.1446), and the correct solution $C = (3, 8.2160)$ now is located slightly *above* the asymptote.

The main point of the illustration above is that the line with slope w/v is the asymptote both for $n \to +0$ and $n \to -0$, in spite of the fact that in the first case distances tend to become infinitely large, whereas in the second case they tend to become infinitely small.

4.4 $\underline{n < 0}$

Figure 9A gives the illustration for $n = -1$. Clearly, this figure is rather similar to that for $n = 0$ (Figure 8). In the latter solution, however, the curved branches form a perfect hyperbola ($xy = vw$), whereas in Figure 9A they do not. It remains valid, though, that in Figure 9A the branches pass through points (v, w) and $(-v, -w)$, and that they have asymptotes at $x = 0$ and $y = 0$. They also pass through the two points (w, v) and $(-w, -v)$ – remember that this is true for any value of n.

The straight line in Figure 8 is replaced by a curved line in Figure 9A. This line still passes through the points (v, w), $(0, 0)$, and $(-v, -w)$. But whereas in Figure 8 the straight line is defined by its constant slope (w/v), the curve is Figure 9A tends to remain close to an asymptote

$$y/x = (w/v)^{1/(1-n)}$$

which is the same as the solution for s in Section 4.2. In Figure 9A, where $n = -1$, it means that the asymptote has slope $3^{1/2} = 1.73$.

Figure 9B gives the illustration for $n \to -\infty$. The curves in Figure 9A now are replaced by broken straight lines. Remember that $n \to -\infty$ implies that the distance between two points depends only upon the horizontal distance when this is smaller than the vertical distance, and depends only upon the vertical distance when this is smaller than the horizontal one. It follows that points $C = (x, y)$ have the same distance to $A = (v, -w)$ and $B = (-v, w)$ when $x = 0$, and either

$$y > (v + w),$$
$$(w - v) > y > (v - w),$$
$$y < -(v + w).$$

This explains the vertical parts of the curves in Figure 9B. They are defined by points $C = (0, y)$ where the horizontal distance between C and A or B is equal: $\underline{AC} = \underline{BC} = v$. In a similar way, for points on the horizontal parts of the curves (where $y = 0$, and where either $x > (v + w)$ or $x < -(v + w)$), the vertical distances to A and B are both equal to w, and smaller than the horizontal distances. Finally, for the oblique lines at the right-upper part of the figure, the horizontal distance to A is equal to the vertical distance to B, whereas for the oblique lines at the lower-left, the vertical distance to A is equal to the horizontal distance to B.

5. Distance d as a function of y for given values of x and n

5.1 Introduction

Let $A = (x, y)$ and $O = (0, 0)$. We want to consider how distance d between A and O depends upon the value of y, for fixed values of x and n.

The following paragraphs give illustrations for various values of n. For convenience of notation, these examples are restricted to positive values of y and x – the reason being that the formula for d always takes the positive absolute values $| x |$ and $| y |$. The illustrations also assume that x has the fixed value $x = 1$.

5.2 n > 0

For positive values of n, the examples are given in the Figures 10A to 10D. Figure 10A shows the curve for $n = 1$. Then $d = x + y = 1 + y$, and the plot becomes a straight line with slope 1, passing through point (1, 0). Clearly the contribution of y to d is equal to y itself, and is independent of the value of x. In other words: there is no interaction between y and x.

Figure 10B shows the graph for $n \rightarrow \infty$. Distance d now depends upon the largest value of x or y. Therefore $d = 1$ as long as $y < 1$, and $d = y$ when $y > 1$.

Figure 10C shows a solution for $1 < n < \infty$, in particular the solution for $n = 2$. Clearly, this curve is bounded by the solutions for $n = 1$ and $n \rightarrow \infty$. In other words, the contribution of y to d remains small as long as $y << x$, whereas the contribution of x becomes negligible when $y >> x$. (At $y = 0$, the curve has zero slope.)

Figure 10D shows a solution for $0 < n < 1$, in particular for $n = .8$. The curve has infinite slope where $y = 0$, but its slope decreases to a value of 1 to the extent that y increases towards $y \rightarrow \infty$. In other words: the contribution of y to d tends to become equal to y itself when y is very large. On the other hand, the contribution of y to d is highly "exaggerated" when y is very small relative to x. Also, the curve for $n < 1$ does not have a finite asymptote with slope 1. (From the formal point of view, its asymptote is: $d = y + \infty$.)

5.3 $n < 0$

When $n < 0$, the curve for d as a function of y has an upper asymptote at $d = x$. This is shown in Figure 10E for $n = -1$.

The limiting case, where $n \rightarrow -\infty$, is given in Figure 10F. Distance d now depends only upon the smallest of x or y, so that $d = y$ when $y < x$, and $d = x$ when $y > x$.

Remember that a negative value n produces the same formula for distances as the application of the positive value $| n |$ to the inverses of d, x, and y. This implies, for instance, that the curve for $n = -1$ in Figure 10E becomes identical to the curve for $n = 1$ (shown in Figure 10A) when results for $1/d$ as ordinate are plotted against $1/y$ as abscissa (given that in the examples it is assumed that $x = 1$, so that also $1/x = 1$). Whereas $n = 1$ implies that distance d, as a measure of dissimilarity, is the sum of the dissimilarities $d = (x + y)$, the choice $n = -1$ means that $1/d$ as a measure of similarity is equal to the sum of similarities: $1/d = (1/x + 1/y)$.

A similar relation can be detected when Figure 10F is compared with Figure 10B.

Also, the extreme choice $n \rightarrow -0$ implies that all distances tend to become equal to zero, so that the curve for d as a function of y becomes the line $d = 0$. Its counterpart is the situation where $n \rightarrow \infty$, so that all distances tend to become infinitely large.

6 Additivity of distances

6.1 Introduction

Without loss of generality, let $A = (-v, -w)$ and $B = (v, w)$. Let $Z = (x, y)$ be a point chosen in such a way that there is additivity of distances:

$$\underline{AZ} + \underline{ZB} = \underline{AB}$$

In general it will be true that such an additivity of distances is valid for any point Z located between A and B on the straight line that connects A and B.
Such a point has coordinates

$$Z = (kv, kw)$$

where $-1 > k > 1$. It then follows that

$$\underline{AB} = \{(2v)^n + (2w)^n\}^{1/n} = 2(v^n + w^n)^{1/n}$$

whereas

$$\underline{AZ} = \{(v + kv)^n + (w + kw)^n\}^{1/n} = (1 + k)(v^n + w^n)^{1/n}$$
$$\underline{ZB} = \{(v - kv)^n + (w - kw)^n\}^{1/n} = (1 - k)(v^n + w^n)^{1/n}$$

so that

$$\underline{AZ} + \underline{ZB} = 2(v^n + w^n)^{1/n} = \underline{AB}$$

whatever the value of n.

6.2 $n > 1$

Figure 11A shows the result for $n > 1$. The result is rather trivial, in the sense that Z must be located on the straight line between A and B.

In fact, Figure 11A shows that the rule of 'triangular inequality' is valid. This rule says that for any triangle AZB it will be true that $\underline{AZ} + \underline{ZB} > \underline{AB}$. The rule is very well-known in Euclidean geometry ($n = 2$). But its validity is more general, and covers all cases when $n > 1$.

6.3 $\underline{n = 1}$

In this case, additivity $\underline{AZ} + \underline{ZB} = \underline{AB}$ is true for all points Z located not outside the rectangle of which A and B are opposite corners (the two other corners are $(v, -w)$ and $(-v, w)$).

The formal proof is easy enough. For any such point $Z = (x, y)$ we have:

$$\underline{AZ} = (v + x) + (w + y)$$
$$\underline{ZB} = (v - w) + (w - y)$$

so that

$$\underline{AZ} + \underline{ZB} = 2v + 2w = \underline{AB}.$$

6.4 $\underline{n \rightarrow \infty}$

This choice of n implies that the distance between points is equal to their "horizontal" distance if it is longer than the vertical one, and equal to their vertical distance otherwise. It implies that $\underline{AZ} + \underline{ZB} = \underline{AB}$ not only for points Z located on the straight line between A and B, but also for all points Z which are located in the interior of the oblique rectangle of which A and B are two corners, whereas the other two corners are given by (w, v) and $(-w, -v)$.

Figure 11B shows this solution as a shaded rectangle, for the example where $v = 1$ and $w = 3$.

6.5 $\underline{0 < n \leq 1}$

This choice of n implies that additivity is not only satisfied by points Z on the straight line between A and B, but also by Z located outside the upright rectangle of which A and B are two corners. This is illustrated in Figure 11C for the choice $n = 1/2$. The figure shows that points Z also may be located on a hexagon, with curved sides, and of which A and B are two of the six corners.

Such a hexagon appears for the following reason. Let Z be one of the other two corners of the upright rectangle, such as $Z = (v, -w)$. Then

$\underline{AZ} = 2v,$
$\underline{ZB} = 2w.$

Take $n = 1/2$. This implies that

$$\underline{AB} = \{(2v)^{1/2} + (2w)^{1/2}\}^2 = 2v + 2w + 2(4vw)^{1/2} > 2v + 2w = \underline{AZ} + \underline{ZB}.$$

This shows that the path from A to B via corner $Z = (v, -w)$ is shorter than the direct path $\underline{AB}$. This implies that there must be paths via points Z which are located *outside* the upright rectangle, in such a way that such paths become *longer* than the path via corner $(v, -w)$, and therefore may become equal to $\underline{AB}$.

In particular, such a point is $Z = (-1, 4.648)$. In the example, with $v = 1$ and $w = 3$, whereas $n = 1/2$, we then obtain

$$\underline{AB} = (2^{1/2} + 6^{1/2})^2 = 14.928$$
$$\underline{AZ} = 3 + 4.648 = 7.648$$
$$\underline{BZ} = (2^{1/2} + 1.468^{1/2})^2 = 7.280$$

so that $\underline{AZ} + \underline{BZ} = \underline{AB}$. This solution for Z gives one of the corners of the hexagon shown in Figure 11C. For reasons of symmetry, the point (1, -4.648) must be another corner. Also, it can be calculated that there are two corners at (2.007, -3) and (-2.007, 3).

It can be shown that the curved hexagon always has two corners at $A = (-v, -w)$ and $B = (v, w)$. There are also two corners at $\pm(v, -s)$ where $w < s < 3w$; and two corners at $\pm(r, -w)$ where $v < r < 3v$.

6.6 $\underline{n = 0}$

We have noted before that $n = 0$ produces "degenerate" results, because it implies that distances between two points become infinitely large when $n \rightarrow +0$, or infinitely small when $n \rightarrow -0$. Nevertheless, the requirement

$$\underline{AZ} + \underline{ZB} = \underline{AB}$$

must have the same limit for solutions of Z, irrespective of whether $n \rightarrow +0$ or $n \rightarrow -0$.

Following the same sort of reasoning as in Section 4.3, it can be shown that

$$p^n + q^n \rightarrow (1 + \alpha) + (1 + \beta) = 2 + \alpha + \beta$$

when $n \rightarrow 0$.

In the equation above, α and ß are very small numbers. It then follows that

$$(1 + \alpha/2)^2 = 1 + \alpha + \alpha^2/4 \cong 1 + \alpha$$

because the value of α^2 is negligibly small. Therefore

$$(1 + \alpha/2) \cong (\mathrm{p}^{1/2})^n.$$

This leads to the result that

$$\begin{aligned}(p^n + q^n)^{1/n} &= (2 + \alpha + ß)^{1/n} \\ &= 2^{1/n}(1 + \alpha/2 + ß/2)^{1/n} \\ &= 2^{1/n}(pq)^{1/2}.\end{aligned}$$

Returning now to the example above, where $A = (-v, -w)$ and $B = (v, w)$, let $Z = (x, y)$ be a point where $x > v$ and $y > w$. The additivity requirement will be satistied when

$$\{(v + x)^n + (y + w)^n\}^{1/n} + \{(v - x)^n + (y - w)^n\}^{1/n} = \{(2v)^n + (2w)^n\}^{1/n}.$$

When $n \rightarrow +0$, the mathematical argument above implies that this equation is equivalent to

$$2^{1/n}(v + w)^{1/2}(y + w)^{1/2} + 2^{1/n}(v - x)^{1/2}(y - w)^{1/2} = 2^{1/n}(2v)^{1/2}(2w)^{1/2}.$$

Dividing by $2^{1/n}$, and squaring both sides, leads to

$$2wx + 2y + 2(v^2 - x^2)^{1/2}(y^2 - w^2)^{1/2} = 4vw$$

or

$$(v^2 - x^2)^{1/2}(y^2 - w^2)^{1/2} = 2vw - vy - wx.$$

Taking squares again results into

$$w^2x^2 + v^2y^2 - v^2\mathrm{w}^2 - x^2y^2 = 4v^2w^2 + w^2x^2 + v^2y^2 - 4v^2wy - 4vw^2x + 2vwxy$$

which can be re-arranged as

$$(xy + vw)^2 + 4vw(vw - wx - vy) = 0.$$

This equation, of 4th degree, determines the right upper curved side of the hexagon in Figure 11D –since in the derivation above it has been assumed that $x > v$ and $y > w$. However, it can be shown that the equation is valid for all points Z located on one of the six sides of the hexagon.

It is readily verified that the equation is valid for solutions

$$Z = \pm(v,\ w)$$
$$Z = \pm(-v,\ 3w)$$
$$Z = \pm(3v,\ -w).$$

These six solutions define the corner points of the hexagon.

Figure 11D also contains the straight line which connects A and B. In fact, points Z located on this line, and between A and B, also obey the additivity requirement that $\underline{AZ} + \underline{ZB} = \underline{AB}$.

6.7 $\underline{n < 0}$

With $n < 0$, points Z for which there is additivity $\underline{AZ} + \underline{ZB} = \underline{AB}$ remain located on a curved hexagon, very similar to that for $n = 0$, and with exactly the same six corners. It also remains valid that such points Z may be located on the straight line between A and B. Figure 11E gives the illustration, for $n = -1$.

6.8 $\underline{n \to -\infty}$

This choice of n implies that the distance between two points is given by their horizontal or by their vertical distance, whichever is the smallest of the two. For the numerical example the result is shown in Figure 11F. Sides of the hexagon no longer are curved. Instead, they are broken straight lines. In Figure 11F, the upper right side of the "hexagon", between the corners $(-v, 3w)$ and (v, w), now is broken into the vertical line between $(-v, 3w)$ and $(-v, w + 2v)$, and the line with slope -1 between $(-v, w + 2v)$ and (v, w).

But there is more. In fact, additivity $\underline{AZ} + \underline{ZB} = \underline{AB}$ is valid for all points Z located in the shaded part of the vertical strip between $x = -v$ and $x = v$. The reason is that the choice $n \to -\infty$ implies that $\underline{AB} = 2v$, whereas for points Z located in the shaded strip we have $\underline{AZ} = (v + x)$ and $\underline{ZB} = (v - x)$, so that $\underline{AZ} + \underline{ZB} = 2v = \underline{AB}$.

7 Interpretation of Minkowski distances

7.1 Assessment of dissimilarities

The general idea is that a subject's assessment of dissimilarity between two objects can be represented in spatial terms. The two objects then appear as points in that space, and the dissimilarity between the two objects is represented by the distance between these two points. If two points *A* and *B* in the spatial picture are close together, this would represent that the two objects *A* and *B* are quite similar.

Assessment of dissimilarity between two objects, however, depends upon a large number of factors. We shall list some of those factors, without pretending that this list is exhaustive.

(a) It makes a difference whether it is *prescribed* to the subject which dimensions must be taken into account, compared to the situation where it is left to the subject himself to select such dimensions.

(b) Suppose the subject is told beforehand which objects he will be asked to compare. It then depends upon the nature of that set of objects whether *A* and *B* are very similar or very dissimilar. For instance, let the set consist of names of composers who lived in the first half of the 20^{th} century. The subject than may judge that Ravel and Strawinsky are very much different. But if the set consists of names of artists (composers, writers, painters, etc.) who lived between 1500 and 1950, then the subject may indicate that Ravel and Strawinsky are very much similar (both two were composers, living during the first half of the 20^{th} century).

(c) Assessment of dissimilarity may not be related to assessment of similarity. When a subject is asked to evaluate the similarity between two objects, he may emphazise the dimension on which the two objects are similar. But asked to evaluate the dissimilarity between the same two objects, the subject may shift to other dimensions. For instance, Debussy and Ravel are quite similar, in a way (both of them French composers, living at the beginning of this century, etc.). On the other hand, they are not similar at all (if you look at their musical style, their orchestration, etc.).

(d) When a subject is asked to make comparisons in succession, there is no guarantee that the subject will remain faithful to the same criteria throughout. The subject may vacillate between emphasis on similarity or on dissimilarity. He may shift in his choice of relevant dimensions. And even if the dimensions are prescribed beforehand, the subject may change his interpretation of those dimensions –or the subject may shift with respect to the amount of weight he gives to the dimensions.

The conclusion from this tentative list of interfering factors, seems to be that a researcher who is looking for just *one* model that explains all responses given by a subject, may neglect the truism that there are more things between heaven and earth than dreamt of in his philosophy.

7.2 Rules

In the following discussion it will be assumed (for simplification of the arguments) that a subject is more or less consistent. In addition, in the examples it will be assumed that the subject makes judgements based upon only two different dimensions. Thirdly, it will be assumed, mainly for convenience, that a measure of similarity is the *inverse* of a measure of dissimilarity, and *vice versa*.

The discussion will be based, mainly, upon the interpretation of the equi-distance contour. As in previous sections of this paper, such a contour is defined here by all points $A = (x, y)$ which have identical distance d from another point $O = (0, 0)$. Moreover, to avoid notational complexity it will be assumed, in general, that x and y are both larger than zero.

(a) *Rule I.* Assume that $x = y$. A subject is said to follow Rule I if distance $\underline{OA} = d$ (as a measure of dissimilarity) remains the same when a (small) increase of x is compensated by an equal decrease of y (or *vice versa*). In other words, at the point A where $x = y$, the equi-distance contour has slope -1.

It may be noted that the last statement is true for all equi-distance contours, whatever the value of the Minkowski parameter n may be. In fact, as long as the contour is a continous curve, its derivative at point (x, y) is equal to $-(x/y)^{n-1}$, so that the slope has value -1 when $x = y$. It may be helpful, therefore, to look at the region around $x = y$ where the slope is not much different from -1: the region where the slope is between $-(1 + \alpha)$ and $-1/(1 + \alpha)$, given that α is some relatively small value.

Clearly, when $n = 1$, the equi-distance contour has slope -1 everywhere (for all values where x and y are both larger than zero). But when n increases, the region becomes smaller. For instance, if we take $\alpha = .1$, then the slope must be within the limits -1.1 and -1/1.1 = -.909. And when we take $n = 3$, the latter limits of the slope will be valid as long as (x/y) remains between the limits $(1.1)^{1/2} = 1.049$ and $(.909)^{1/2} = .953$. With $n \rightarrow \infty$ the region disappears altogether (the equi-distance contour then becomes an upright square, and has slope 0 when $y > x$, and slope $-\infty$ when $x > y$).

Similary when n decreases from 1 to 0. For instance, when $n = 1/2$, the region is limited by $(.909)^2 < (x/y) < (1.1)^2$, or $.826 < (x/y) < 1.21$. In fact, when $n \to 0$, the region narrows down to $.909 < (x/y) < 1.1$.

At the same time it should be noticed that when $n \to 0$, the equidistance contour becomes a very narrow star. As an illustration, take $n = .1$. The equi-distance contour then has slope -1.1 when $(x/y)^{.9} = 1.1$, or $(x/y) = 1.1117$. At the same time it must be true that $x^{.1} + y^{.1} = d^{.1}$. If we now take the equi-distance contour for $d = 1$, the solution becomes $x =$ 1.0230E-3 and $y =$ .9261E-3. Similarly, the contour has slope -1/1.1 when $x =$ .9291E-3 and $y =$ 1.0295E-3. It thus follows that the equi-distance contour (which is a rather narrow star) has slope between -1.1 and -1/1.1 when x or y is between the limits .9291E-3 and 1.0295E-3. In fact, this is a very narrow interval around the values $x = y =$ .9766E-3, where the slope is equal to -1.

A similar reasoning applies when $n < 0$. For instance, with $n = -1$, the region where the contour has slope between -1.1 and -1/1.1 is defined by $.953 < x/y < 1.049$. The region becomes infinitely narrow when $n \to -\infty$.

Two remarks may be added.

(i) It can be seen above that the boundaries for the region for $n = 3$ are the same as those for $n = -1$. Why?

Let n be some positive number, and m some negative number. It we take a boundary for the slope at -1.1, it follows that we have

$$(x/y)^{n-1} = 1.1$$

and also that

$$(x/y)^{m-1} = 1/1.1$$

so that $(n - 1) = -(m - 1)$, and therefore $n + m = 2$. It thus follows that with $n = 3$ we find the same boundary as with $m = -1$.

(ii) Suppose that the subject applies Rule I to *similarities*. It then follows that Rule I cannot be valid for dissimilarities (and *vice versa*).

This is illustrated in the following example. Suppose $x = 3$, and $y = 1$. The similarities with respect to point (0, 0) then are $1/x = .333$ and $1/y = 1$. Let $1/x$ increase to .433, whereas $1/y$ decreases to .9. These two changes have zero sum, so that $1/d$ does not change. On the other hand, dissimilarity x changes to $1/.433 = 2.309$. This means a change of $3 - 2.309 = .691$. The change in y is equal to $1 - 1/.9 = -.111$. It shows that the decrease in x is much larger than the increase of y. The conclusion is: equal changes in similarity do not correspond to equal changes in dissimilarity.

This effect is less outspoken when $x \cong y$. Let $x = y = 2$, so that $1/x = 1/y = .5$. Let $1/x$ increase to .6, so that x becomes 1.667: a change of -.333. Let $1/y$ decrease with the same value .1, so that $1/y$ becomes .4, and y becomes equal to 2.5. This means a change in y equal to .5.

The conclusion is rather simple. It says that according to Rule I a change from x to $x + \alpha$ is compensated by a change of y to $y - \alpha$: the changes in dissimilarity must have the same absolute value, and must have opposite sign. Changes in similarity then also will have opposite sign, but their absolute value will be more or less equal only if $x \cong y$, and when α is relatively very small.

(b) *Rule II.* Rule II says that the subject, when asked to evaluate dissimilarities, tends to give more weight to the largest of x and y.

In an absolute sense, Rule II means that $d = x$ when $x > y$, and $d = y$ when $y > x$. This implies, in a way, that the subject simplifies his task by reducing it to a one-dimensional task. The subject decides that he will take only one dimension into account, and then selects x (ignoring y) when $x > y$, or selects y (ignoring x) when $y > x$.

If the subject adheres to Rule II, it follows that his responses should match with a spatial representation where $n \rightarrow \infty$. On the other hand, when a subject remains strictly faithful to Rule I, his responses should fit with a spatial representation in which $n = 1$.

However, a subject may compromise between the two rules, in the sense that he applies Rule I when $x \cong y$, and Rule II when $x >> y$ or $x << y$. Such a compromise will tend to match with a representation where $1 < n < \infty$. The best match might be found by taking $n = 2$. However, this does not necessarily mean that the subject has such a mathematical model with $n = 2$ in mind (as if this model "explains" his responses). What the subject has in mind is just some compromise between Rules I and II. Parameter n is no more than an indicator of the extent to which the subject gives dominance to one of the two rules: if Rule I is dominant, then his responses will fit with a description where n is close to 1. But when Rule II is more dominant, a larger value of n will serve better.

It can be said, in a way, that both Rule I and Rule II tend to facilitate the subject's task by making it one-dimensional instead of two-dimensional. Rule I says: look first at x (one-dimensional) and then look at y (also one-dimensional) and let the final response depend upon the sum of those two estimates of dissimilarity: $d = x + y$. Rule II says: first decide on which dimension the dissimilarity is largest, and let the response depend on that dimension, with the other dimension ignored. If the responses fit best with a spatial representation where $1 > n > \infty$, then this just indicates whether the compromise gives relatively more weight to Rule I (n is close to 1), or to Rule II (n becomes large).

(c) *Corollaries: Rules I' and II'.* It has been said before that Rule I also might be applied to similarities, instead of to dissimilarities. If a subject keeps strictly to that rule, responses will match with a Minkowski parameter equal to -1. This corollary of the rule may be called Rule I'.

In the same way Rule II has its corollary Rule II'. It means that the assessment of $(1/d)$ is based only upon the one dimension where similarity is largest, so that the dimension where dissimilarity is largest is ignored. In an absolute sense, Rule II' will produce responses which match with Minkowski parameter $n = -\infty$.

But, again, the subject may compromise, in the sense that Rule I' is more or less followed when $(1/x) \cong (1/y)$, whereas Rule II' is taken when there is a large difference between $(1/x)$ and $(1/y)$ –in particular when one of the two similarities becomes close to zero. Such a compromise will tend to match with a spatial representation where $-1 < n < -\infty$.

(d) *Rule III.* Thus far we have proposed rules which will tend to match with a spatial representation in terms of a Minkowski model where $n > 1$, or $n < -1$. Rule III (and its corollary Rule III') is meant to explain why it may happen that a model with $-1 < n < 1$ seems to fit the data.

In its extreme form, Rule III would match with $n \to +0$. It can be argued that this rule implies that $d = x$ when $y = 0$, and $d = y$ when $x = 0$. However, if both x and y are different from zero, the rules implies that $d \to \infty$. One could say: if two objects differ in more than one dimension then the subject will conclude that these two objects are infinitely dissimilar: such objects are unique, incomparable. Or: a comparison between two objects makes sense only if you focus on just one dimension, and ignore all other dimensions.

To give an example: two children are unique human beings, incomparable. Nevertheless, you may compare them if you take just one dimension (such as 'age') and ignore all other dimensions.

The corollary is Rule III'. This rule matches with $n \to -0$. In fact, this rule implies that $d = 0$ for all pairs of objects which differ in more than one dimension. Or: all objects are the same (unless you focus on just one dimension). In the example: all two children are essentially the same: they are just "human beings". Nevertheless, you may notice that they differ in age. But a difference in age does not characterize their status as "human beings".

Rule III is, in my point of view, related to a rather extreme, perhaps absurd, philosophy, which maintains that "everything is unique". Rule III' refers to an equally extreme opposite philosophy: "all things are the same". It is interesting that two such extremely opposite points of view can be related to an infinitely small difference in the choice of the Minkowski parameter n: either $n \rightarrow +0$ or $n \rightarrow -0$. (One could imagine a philosopher who argues that all things are the same because they share the property of being unique.)

A more moderate approach towards an interpretation of Rule III would be to look at *relative* changes. Let x change to $(x + \alpha)$. It is reasonable to say that this change (for a given value of α) becomes less important to the extent that x is larger. For instance, a change from $x = .25$ to $(x + \alpha) = .50$ seems more important than the (opposite) change from $y = 6.25$ to $(y - \alpha) = 6$. This becomes more convincing if one looks at the relative change. For x such a relative change can be defined as $\alpha/x = .25/.25 = 1$, or as $\alpha/(x + \alpha) = .25/.50 = .50$. It seems less ambiguous to define the relative change with respect to the average of x and $(x + \alpha)$, so that it becomes equal to $2\alpha/(2x + \alpha) = .5/.75 = .667$.

Similarly, in the example above the relative change for y can be defined as $2\alpha/(2y - \alpha) = .5/12.25 = .0408$. Clearly, although the absolute value of the changes in x and y is the same, the relative changes are quite different.

One advantage of relative change is that it has the same value both for dissimilarities and similarities. In the example above, we have similarity $1/x = 4$, changed to $1/(x + \alpha) = 2$. The absolute value of this change is equal to 2, but the relative value becomes $2/3 = .667$, which is the same as the relative change in dissimilarity. Similarly, $1/y = .16$ changes to $1/(y - \alpha) = .1667$ so that relative change is equal to $.0067/.1633 = .0408$.

The argument above implies that Rule I has the effect that a large difference between x and y goes together with a large difference in relative changes. Now it may quite well be that a subject, asked to assess equi-distances, does not accept such discrepancies in relative change. The subject then may give responses such that the discrepancy in relative changes becomes smaller. His responses then will tend to match with a Minkowski model where $-1 < n < 1$. This is numerically illustrated in Table 2.

This table is based upon the same example as above. The numerical column at the left shows that it is given that $x = .25$ and $\alpha = .25$, so that $(x + \alpha) = .50$. Whatever the value of n, relative change in x is the same as relative change in $1/x$: both relative changes are equal to .667.

The following colums refer to what happens to the given value of $y = 6.25$, if equi-distance to (0, 0) is required, depending upon the choice of n (given in row 13). The column where $n = 1$ corresponds with Rule I. It shows that the absolute change in dissimilarity (row 3) is the same for x and for y. The column where $n = -1$ deals with the case where the absolute changes in similarity are the same (both equal to 2, as shown in row 6).

Tabel 2 also shows that relative change in dissimilarity is always the same as the relative change in similarity (row 9 is equal to row 12).

The last column of the table shows results when $n \rightarrow 0$. Relative change in y now becomes equal to relative change in x (both equal to .667).

Also, when $n \rightarrow +0$, the value of d increases, to $d = \infty$ when $n = +0$. On the other hand, when $n \rightarrow -0$, the value of d becomes smaller, with limit $d = 0$ for $n = -0$. Incidentally, it can be proved that the two values for d at a given value of n and its negative $-n$, have product equal to xy. In the table: (6.50)(.240) = (27.42)(.057) = (153.9)(.010) = xy = (.25)(6.25) = 1.5625.

(The solution for $n = 0$ in Table 2 also must have the property that $\beta/y = \alpha/(x + \alpha)$. In the numerical example both ratios are equal to .5. However, this particular numerical example might, erroneously, suggest that it must generally be true that from $\alpha = x$ it follows that $\beta = (y - \beta)$.)

The general conclusion would be that a subject whose responses match with a Minkowski model where $0 < n < 1$, compromises between Rule I (where $n = 1$) and Rule III (where $n \rightarrow +0$). Similarly, when the responses match with a model where $-1 < n < 0$, the compromise is between Rule I' (where $n = -1$) and Rule III' (where $n \rightarrow -0$). In both cases, the subject tries to reduce the discrepancy between the relative changes in x and y (or their inverses $1/x$ and $1/y$).

7.3 General conclusion

The preceding paragraphs of Section 7 want to make it acceptable that assessment of dissimilarity or similarity between objects may depend upon some rather "intuitive" rules. Rules I, II, III (or their corrolaries) could be such rules, and a subject may compromise between those rules.

It may happen that the subject's application of such rules has the result that his responses tend to match with the Minkowski model (for some selected value of parameter n). However, this model as such does not give an *explanation* of the subject's behavior. The Minkowski model only gives a (more or less) faithful *description* of the responses. What is needed, then, is to explain *why* a Minkowski model gives a good description. The intuitive rules mentioned above are no more than an *attempt* to explain why a Minkowski model may sometimes fit. Empirical research may show that other intuitive rules are much more promising.

APPENDIX

It is well-known that for any triangle in Euclidean space ($n = 2$), there will be only *one* solution for a point Z at equal distance from each of the three corners of the triangle.

In fact, this will be true for any value $n > 1$. It is tempting to assume that it also will be true for $0 > n > 1$.

However, this assumption is not valid. Figure 12 gives an illustration for $n = .5$, where the triangle is formed by the points $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$. In this figure, points at equal distance from A and B are located on the curves ++++, those at equal distance from A and C on curves ------, and those at equal distance from B and C on curves ....

Whenever two of the equi-distance curves intersect at a point Z, the third curve also must pass through this point (because if $\underline{AZ} = \underline{BZ}$ and $\underline{BZ} = \underline{CZ}$, then it follows that $\underline{AZ} = \underline{CZ}$). Figure 12 shows that there are *five* solutions for Z. Their coordinates are listed in Table 3, together with the value of distance d between Z and each of the three corners.

Given one of the solutions for Z, one might draw in Figure 12 a "star" around Z. The size of such a star depends upon the value of d. For all five solutions it must be true that the points A, B, and C are located on the curved sides of the star. In fact, for Z_1 and Z_3 it can be verified that the corners of the triangle are located on three different curved sides of the star. Solution Z_2 implies that B and C are located on the same curved side of the star around Z_2, whereas for Z_4 the points A and B are on the same side, and for Z_5 the same side of the star passes through A and C.

One should not conclude from this example that there always must be five solutions for Z. For instance, when $A = (1, 16)$, $B = (9, 4)$ and $C = (2, 12.858)$ (where $12.858 = (5 - 2^{1/2})^2$), the reader may verify that there are 7 solutions for Z (one of them being $Z = (0, 0)$ with $d = 25$, where the corners A, B, and C are located on the same curved side of the star around Z).

It can be shown, perhaps, that the number of possible solutions for Z does not only depend upon characteristics of triangle ABC, but also upon the apriori choice of a value $n < 1$.

TABLES

Table 1 Table 1A gives the coordinates of five points (*A* to *E*), pictured in Figure 3. Table 1B shows in the right-upper half the city-block distances (n = 1) between these points, and in the left-lower half their corresponding rank numbers. Similarly in Table 1C for Euclidean distances (n = 2).

1A

	A	*B*	*C*	*D*	*E*
x	0	6	5	.6	5
y	2	0	10	7.4	4.1

1B

	A	*B*	*C*	*D*	*E*
A		8	13	6	7.1
B	7		11	12.8	5.1
C	10	8		7	5.9
D	3	9	4		7.7
E	5	1	2	6	

1C

	A	*B*	*C*	*D*	*E*
A		6.32	9.43	5.43	5.42
B	7		10.05	9.16	4.22
C	9	10		5.11	5.90
D	4	8	2		5.50
E	3	1	6	5	

Table 2 Results when it is required that the distance between (0, 0) and (x, y) must be equal to that between (0, 0) and {($x + \alpha$), ($y - ß$)}, where the solution for ß depends upon the choice of parameter n. Data in the first numerical column depend upon fixed values $x = .25$ and $(x + \alpha) = .50$. Data in the other columns depend upon fixed value $y = 6.25$ and upon the selected value of n, given in row 13. Further explanation in the text of Section 7.2(d).

(1) x or y	.25	6.25	6.25	6.25	6.25	6.25	6.25	6.25
(2) x+α or y–ß	.50	6	4.388	3.919	.463	1.850	2.323	3.125
(3) abs.diff.	.25	.25	1.862	2.331	5.787	4.400	3.927	3.125
(4) inv.of row 1	4	.16	.16	.16	.16	.16	.16	.16
(5) inv.of row 2	2	.167	.228	.255	2.16	.541	.431	.32
(6) abs.diff.	2	.007	.068	.095	2	.381	.271	.16
(7) aver.diss.	.375	6.125	5.319	5.085	3.356	4.050	4.287	4.6875
(8) abs.diff.	.25	.25	1.862	2.331	5.787	4.400	3.927	3.125
(9) rel.diff.	.667	.041	.350	.458	1.724	1.087	.916	.667
(10) aver.sim.	3	.163	.194	.207	1.16	.350	.295	.24
(11) abs.diff.	2	.007	.068	.095	2	.381	.271	.16
(12) rel.diff.	.667	.041	.350	.458	1.724	1.087	.916	.667
(13) n		1	.25	.15	-1	-.25	-.15	→ ±0
(14) d		6.50	27.42	153.9	.240	.057	.010	∞ or 0

Table 3 Coordinates of five solutions for *Z*, such that *Z* has the same distance *d* (based upon $n = .5$) to each of the given points *A*, *B*, and *C*. Illustrated in Figure 12.

	x	y	d
A	0	0	
B	3	5	
C	5	2	
Z_1	3.1904	.5699	6.4571
Z_2	2.7016	.8726	6.6451
Z_3	1.2937	2.7666	7.8446
Z_4	3.3942	-2.1060	10.8475
Z_5	-10.4867	3.1799	25.2185

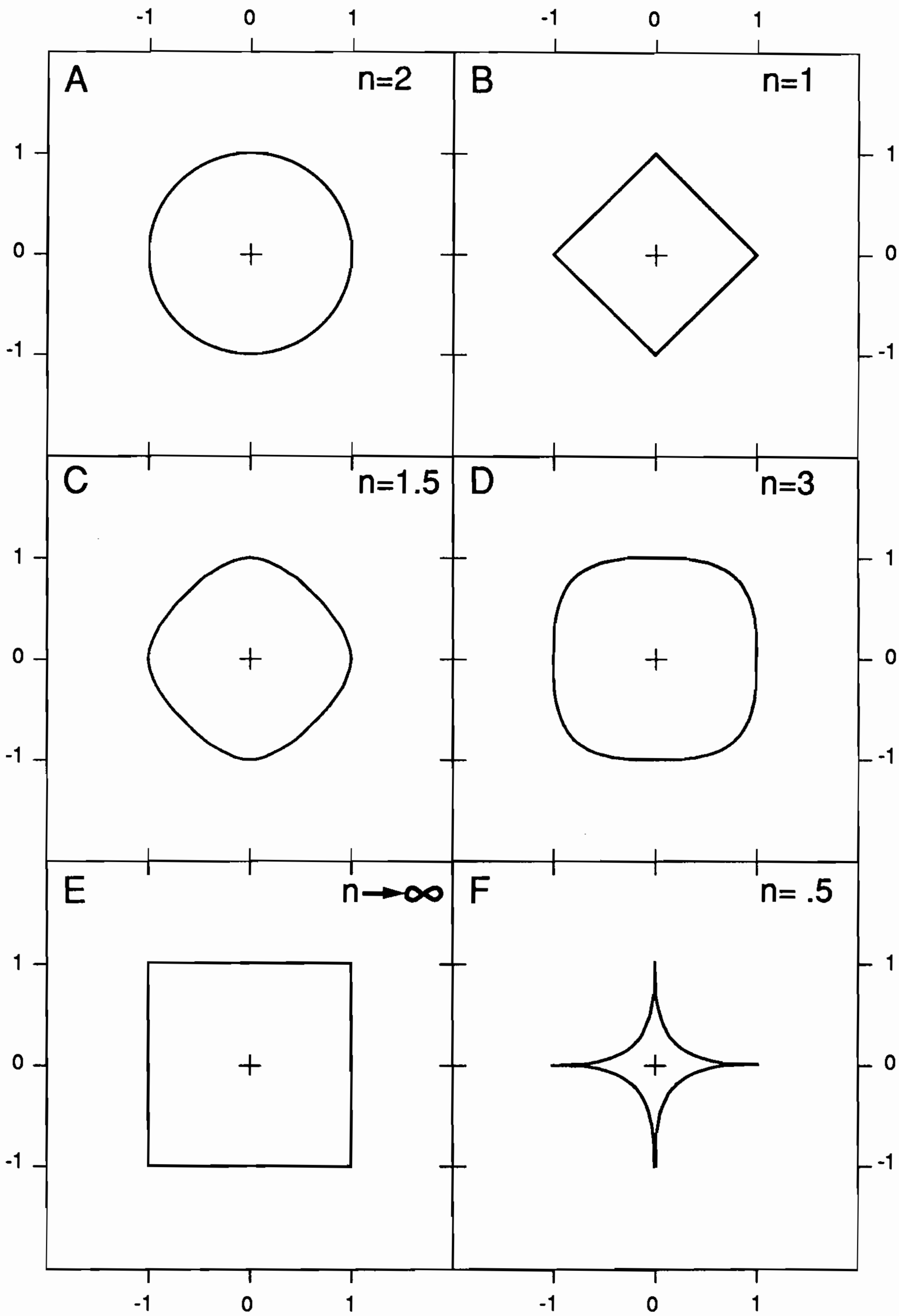


Figure 1A - 1F

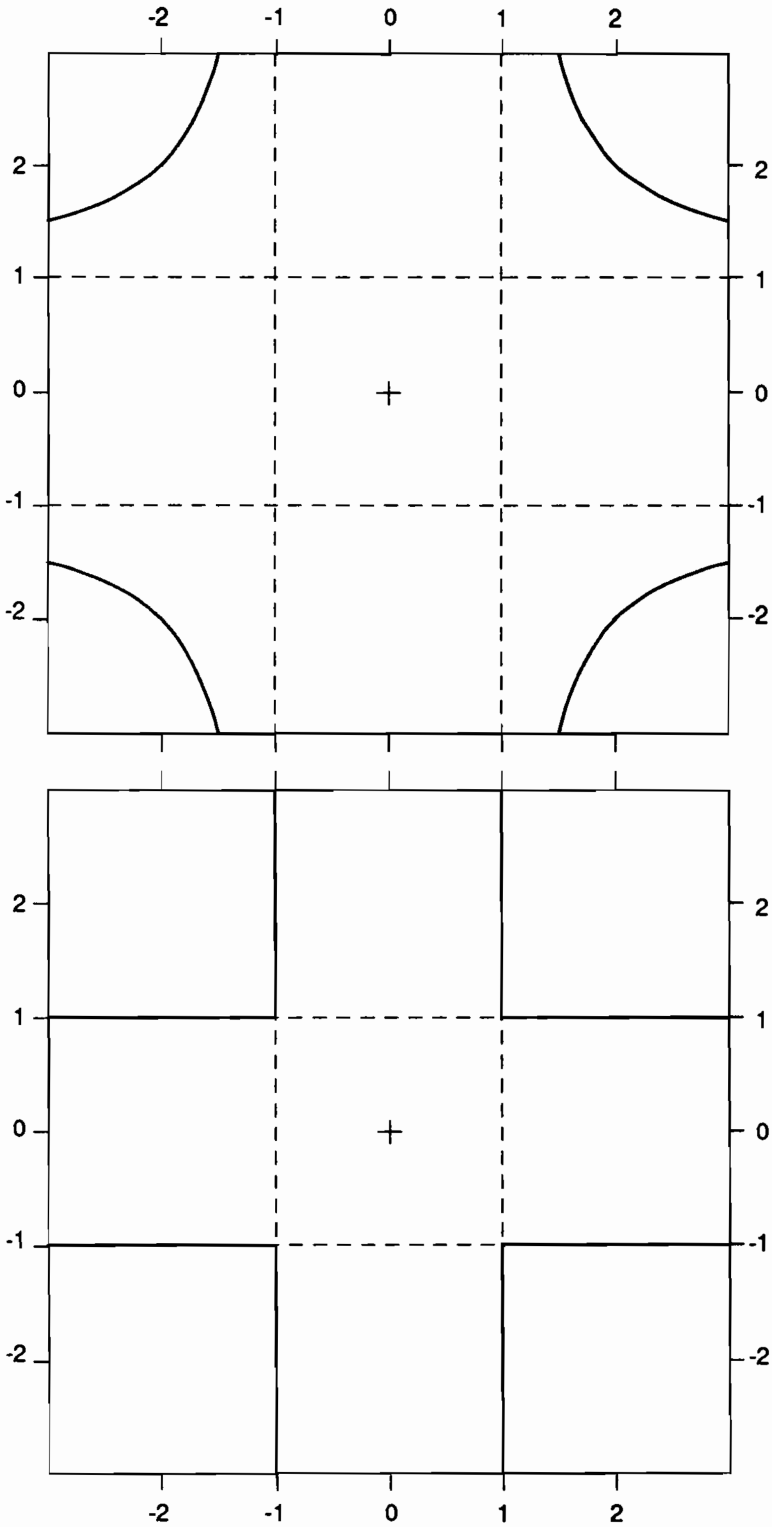


Figure 1G and 1H

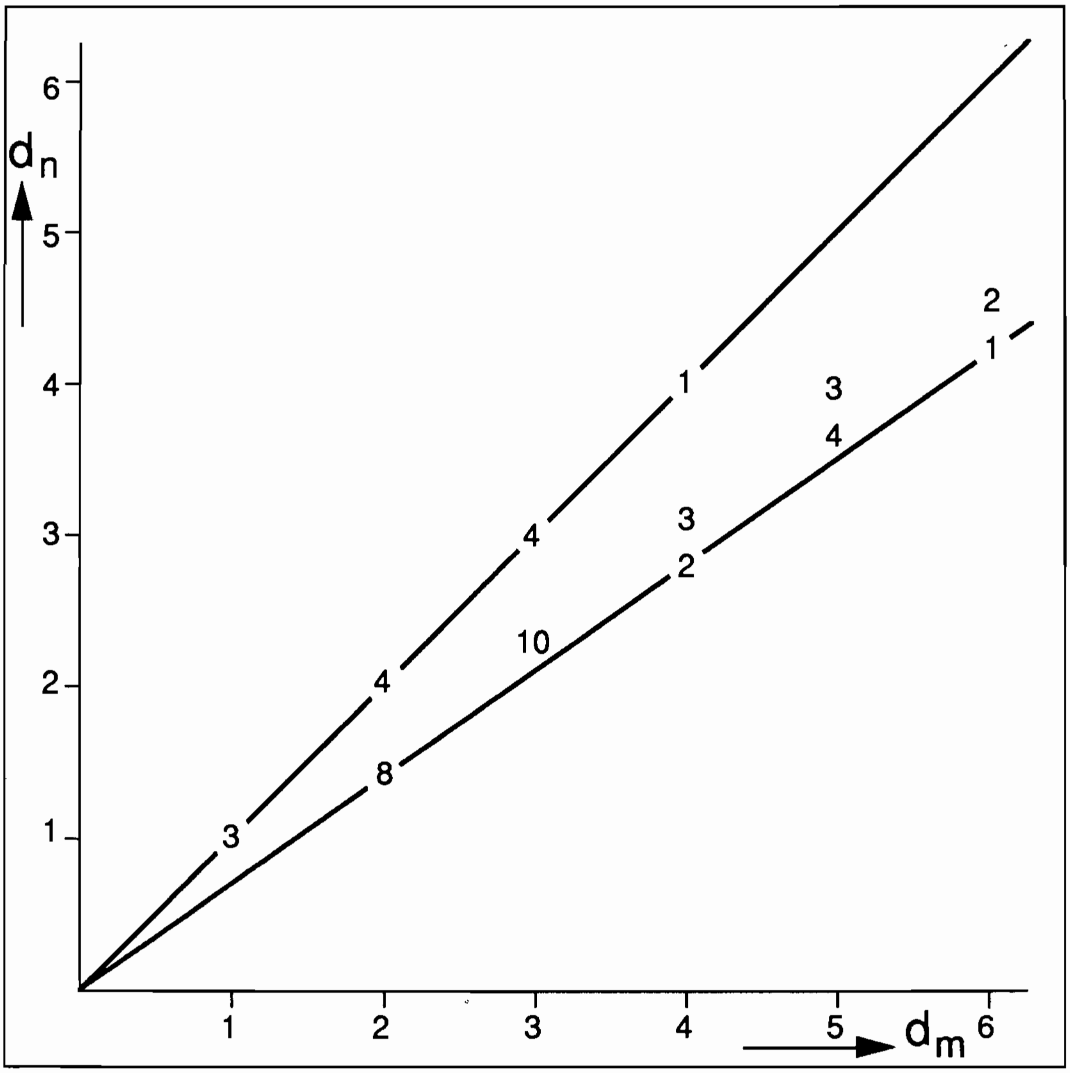


Figure 2

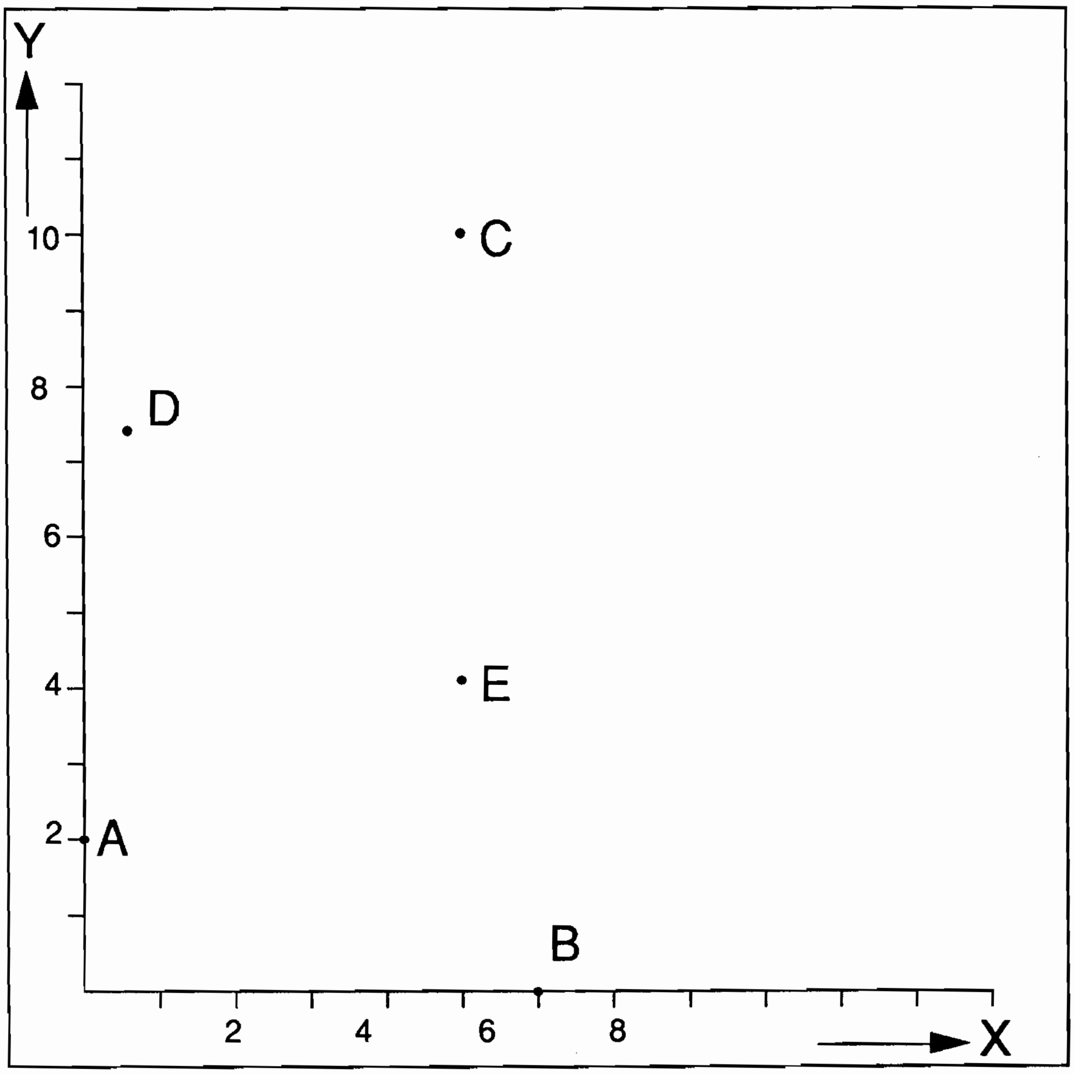


Figure 3

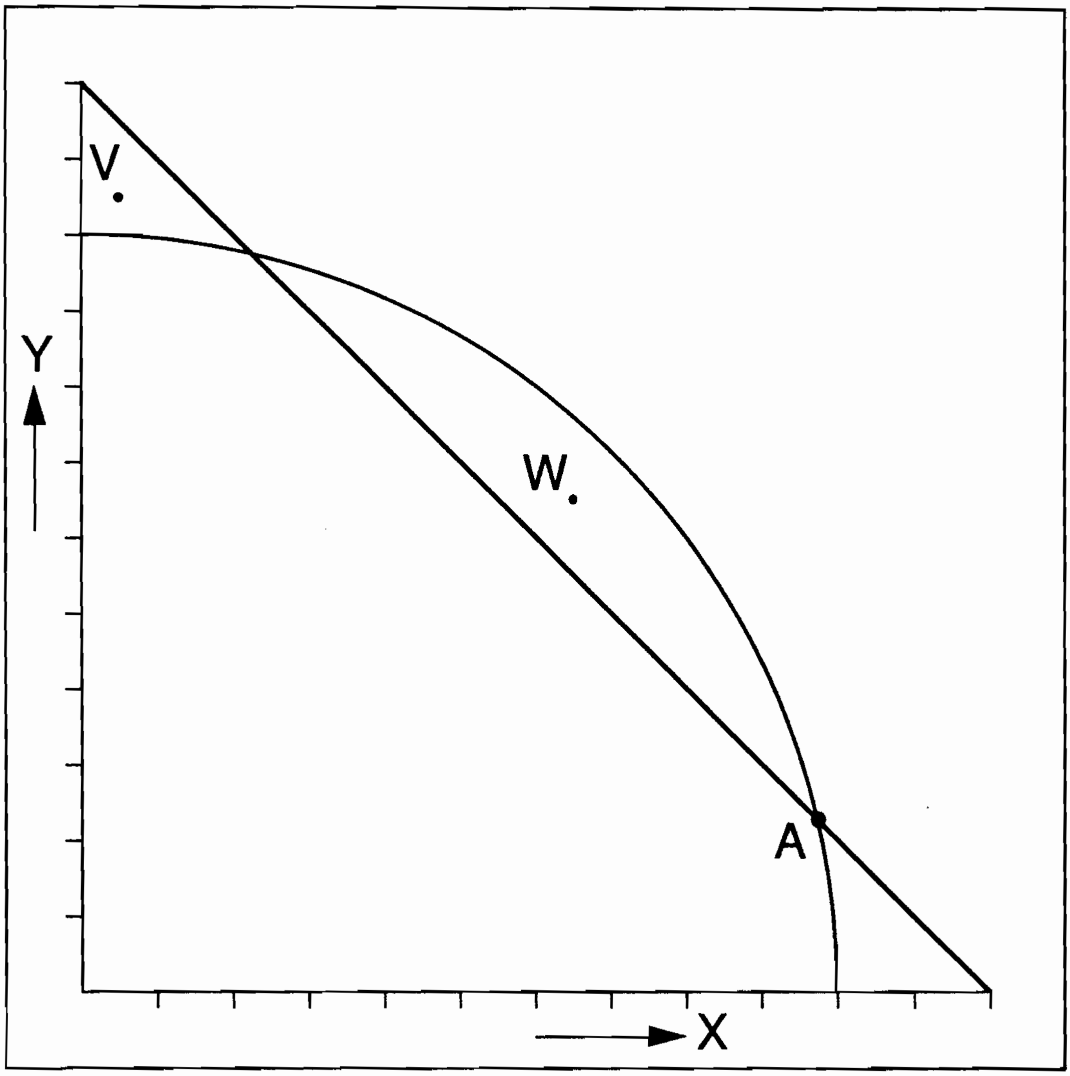


Figure 4

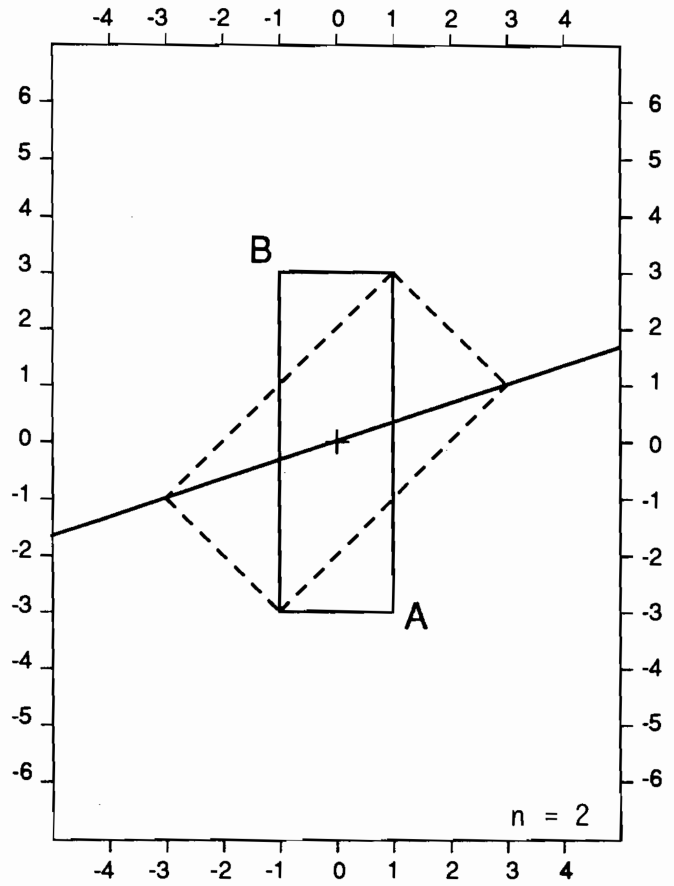


Figure 5A

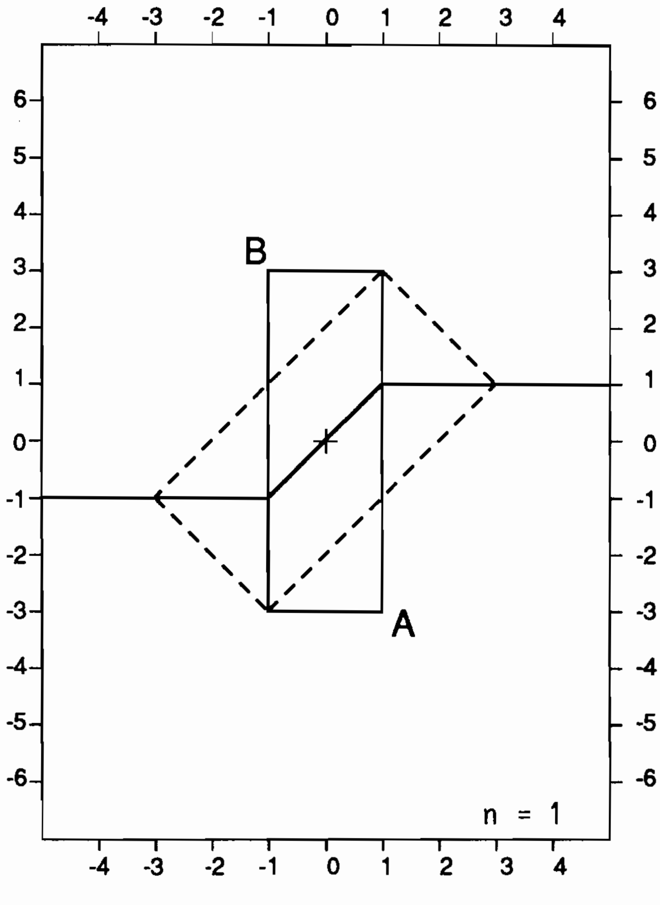


Figure 5B

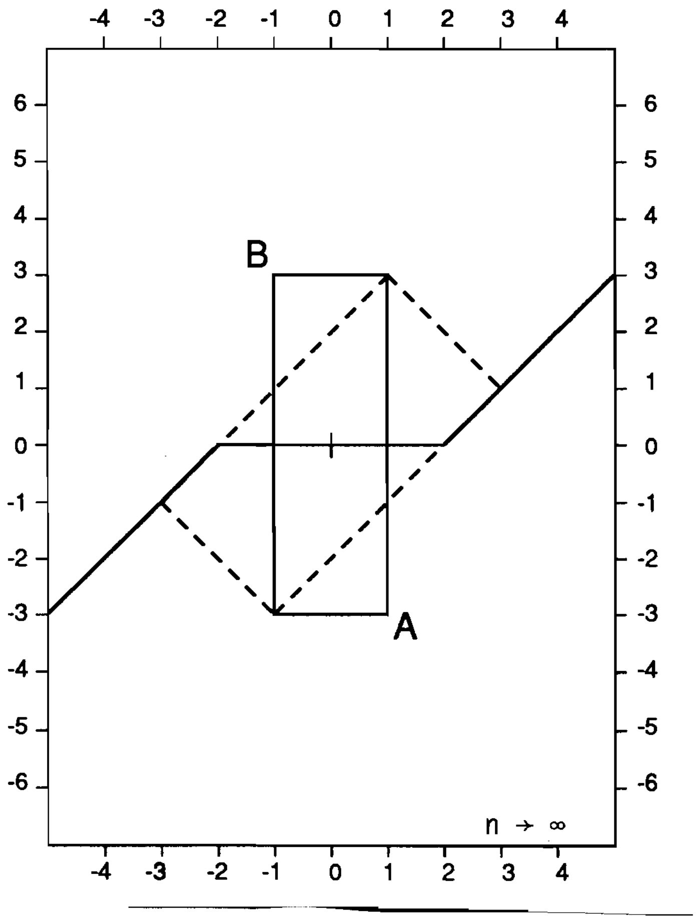


Figure 5C

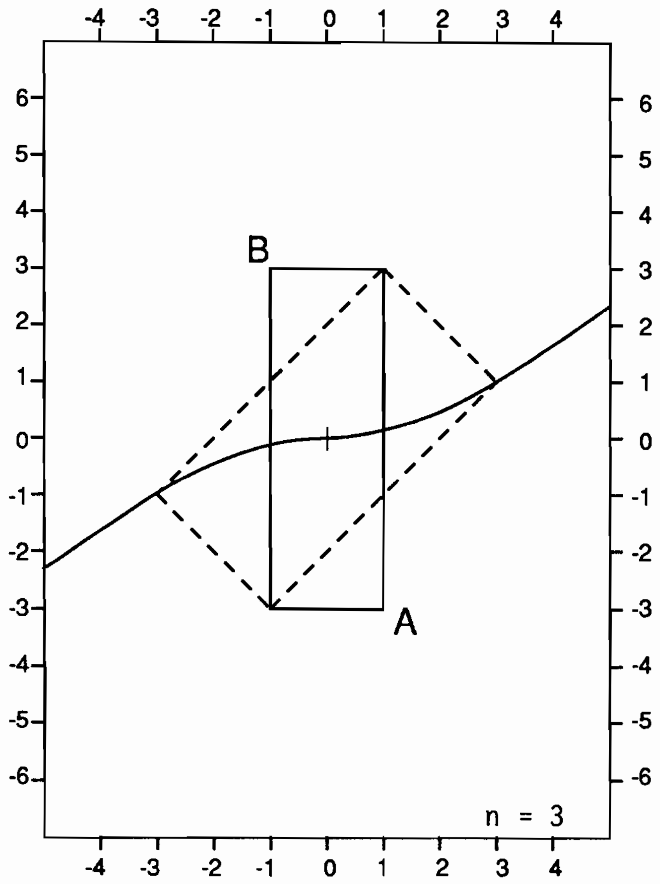


Figure 5D

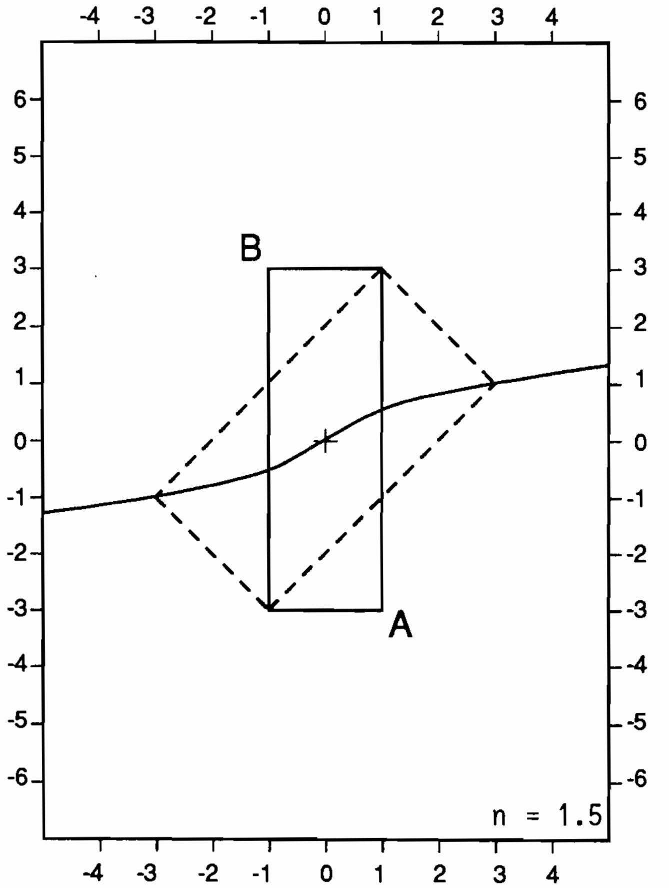


Figure 5E

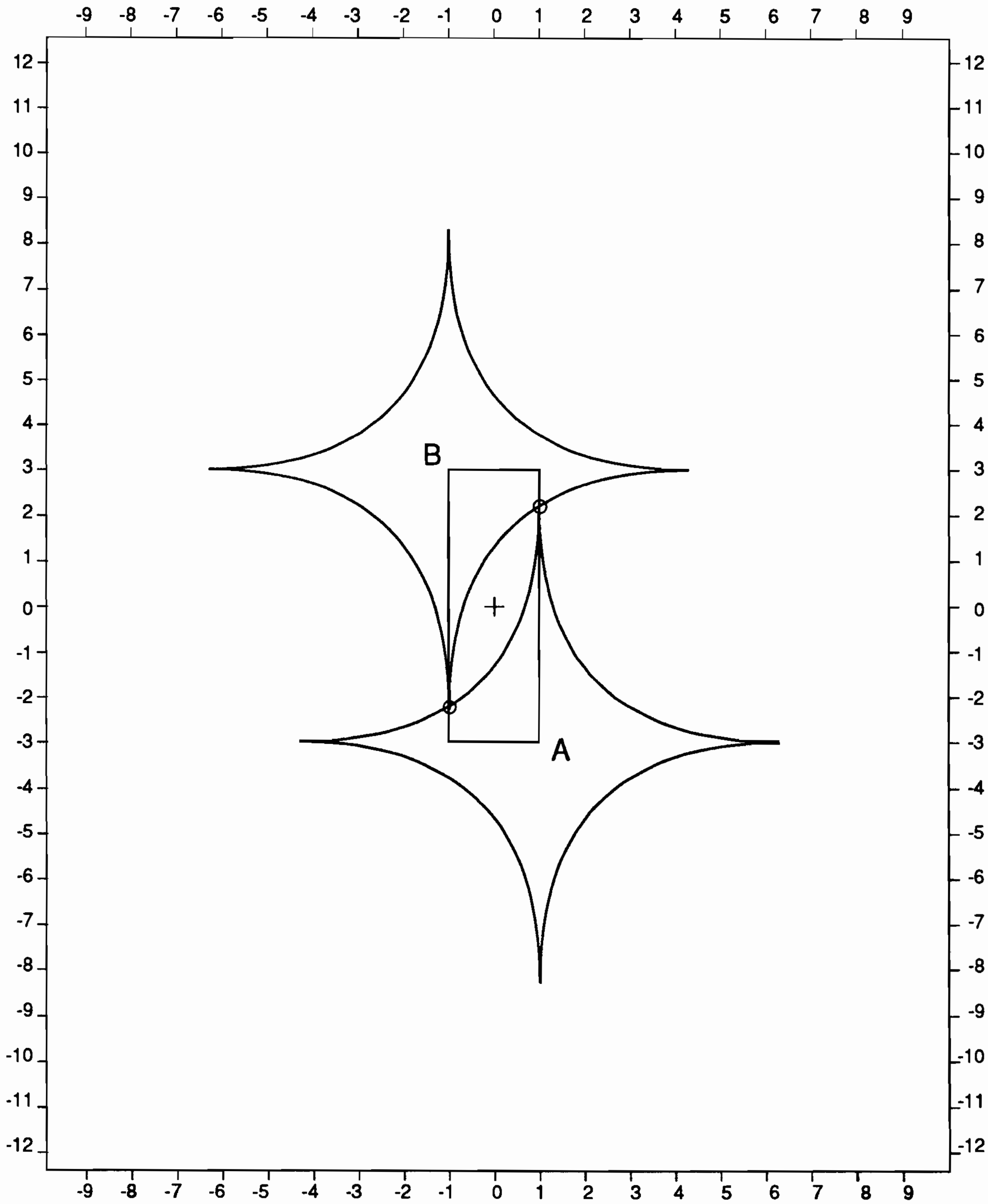
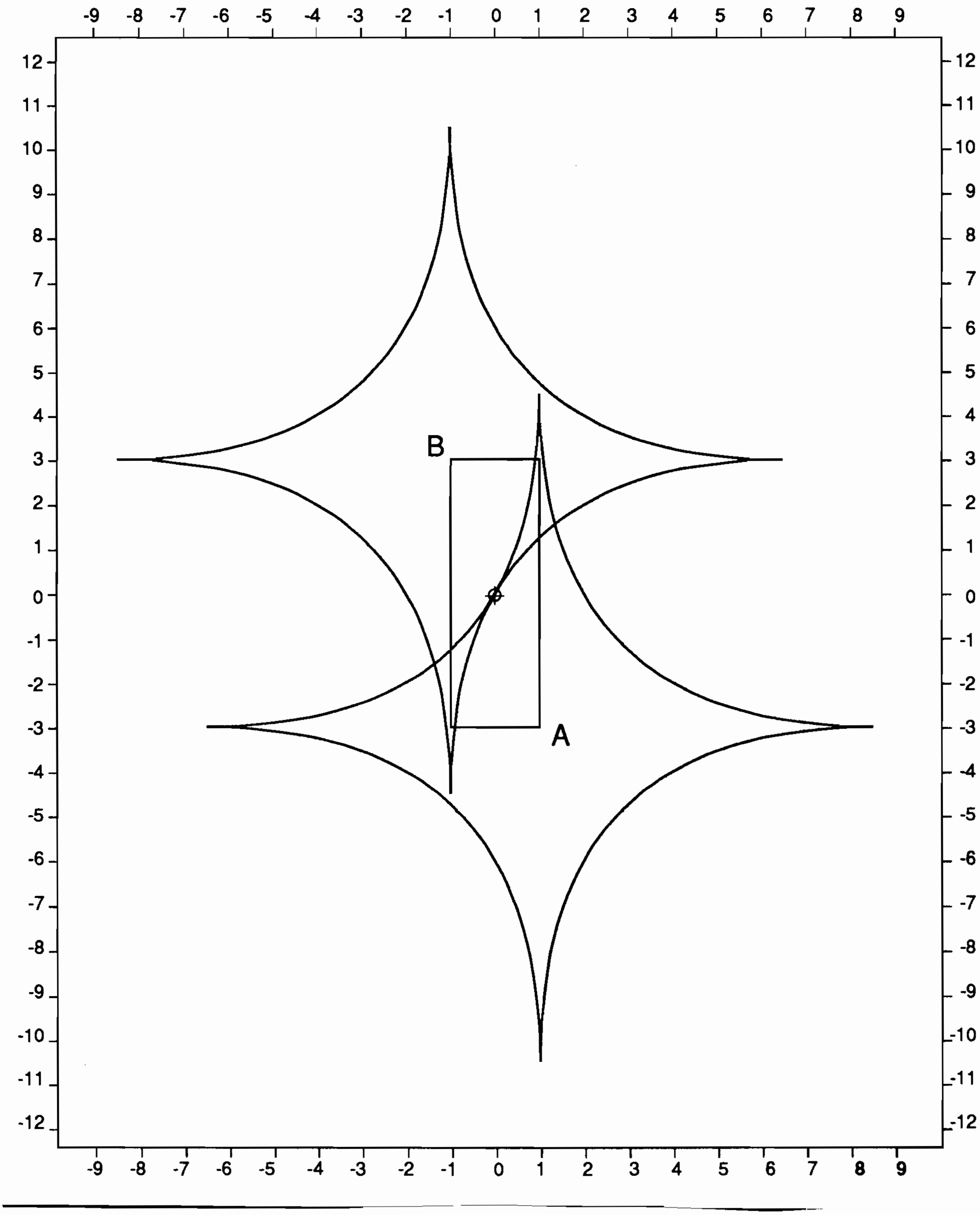


Figure 6A


B
A
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-8
-7
-6
-5
-4
-3
-2
-1
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12
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3
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-5
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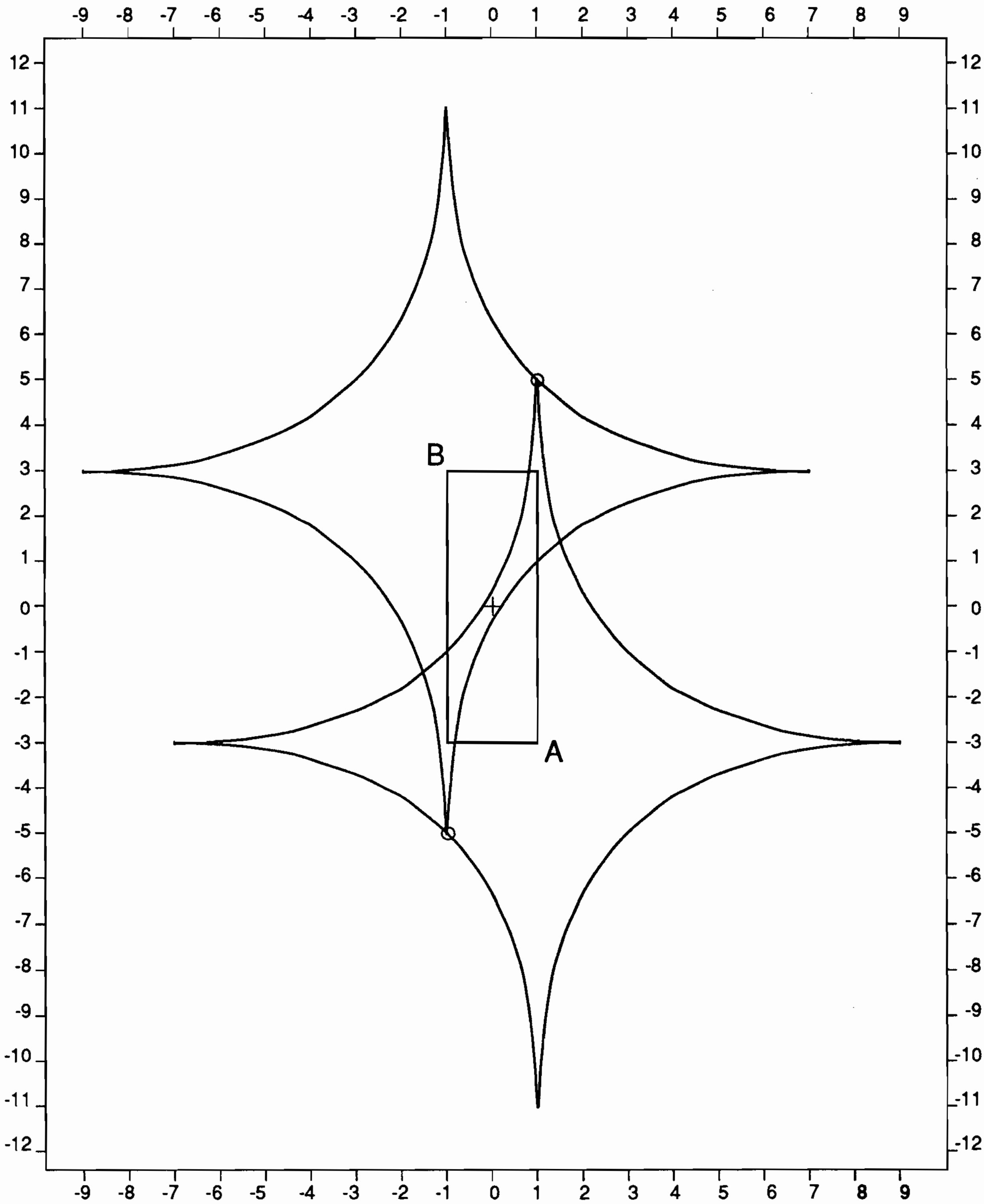
-9
-10
-11
-12

Figure 6B



-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
12 11 10 9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12
B
A

Figure 6C

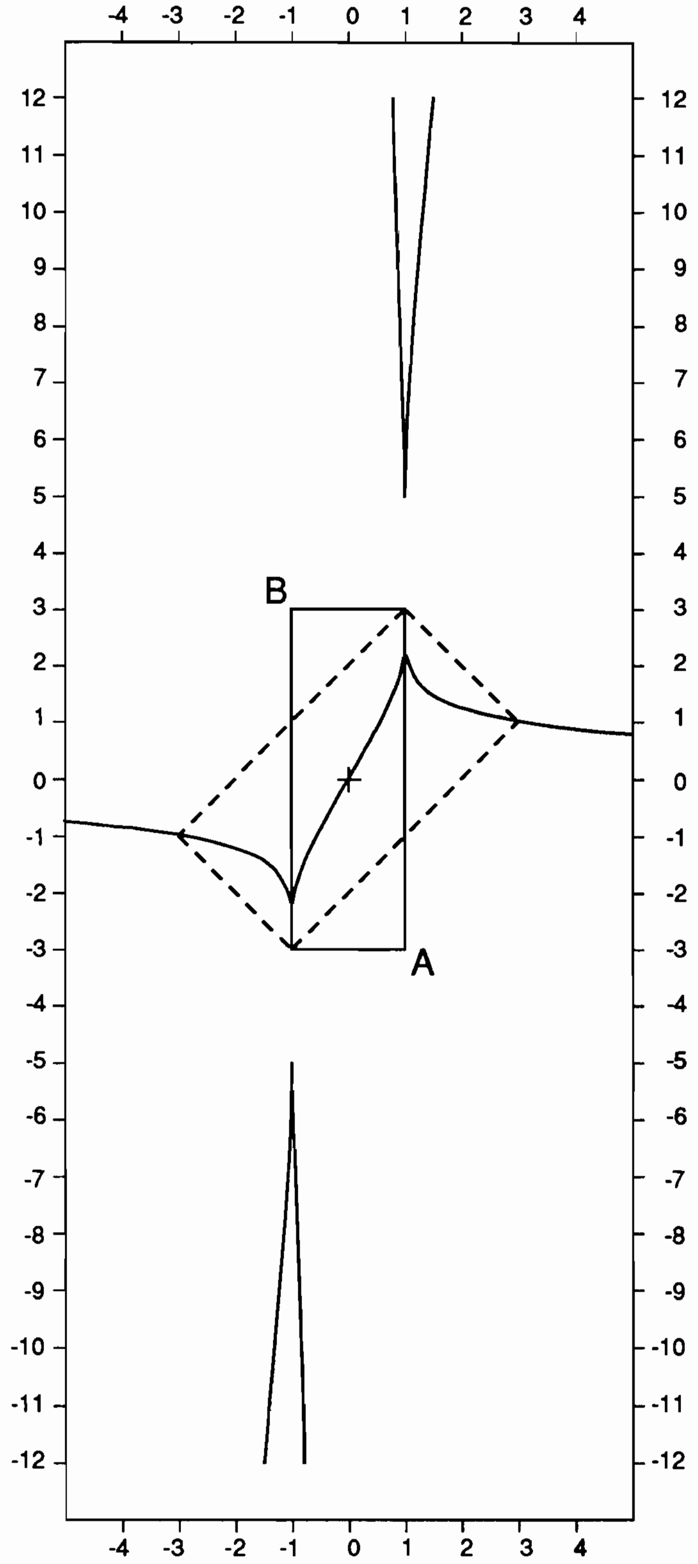


Figure 7

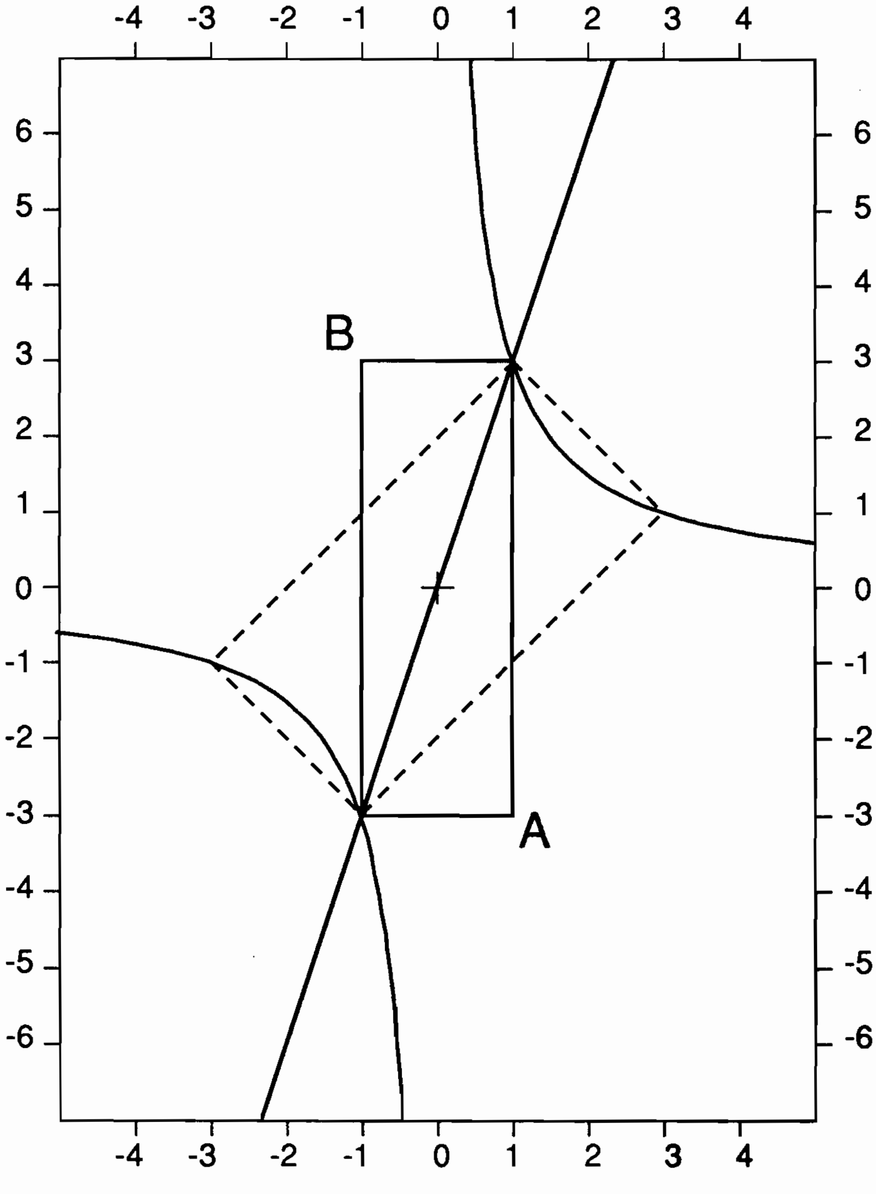


Figure 8

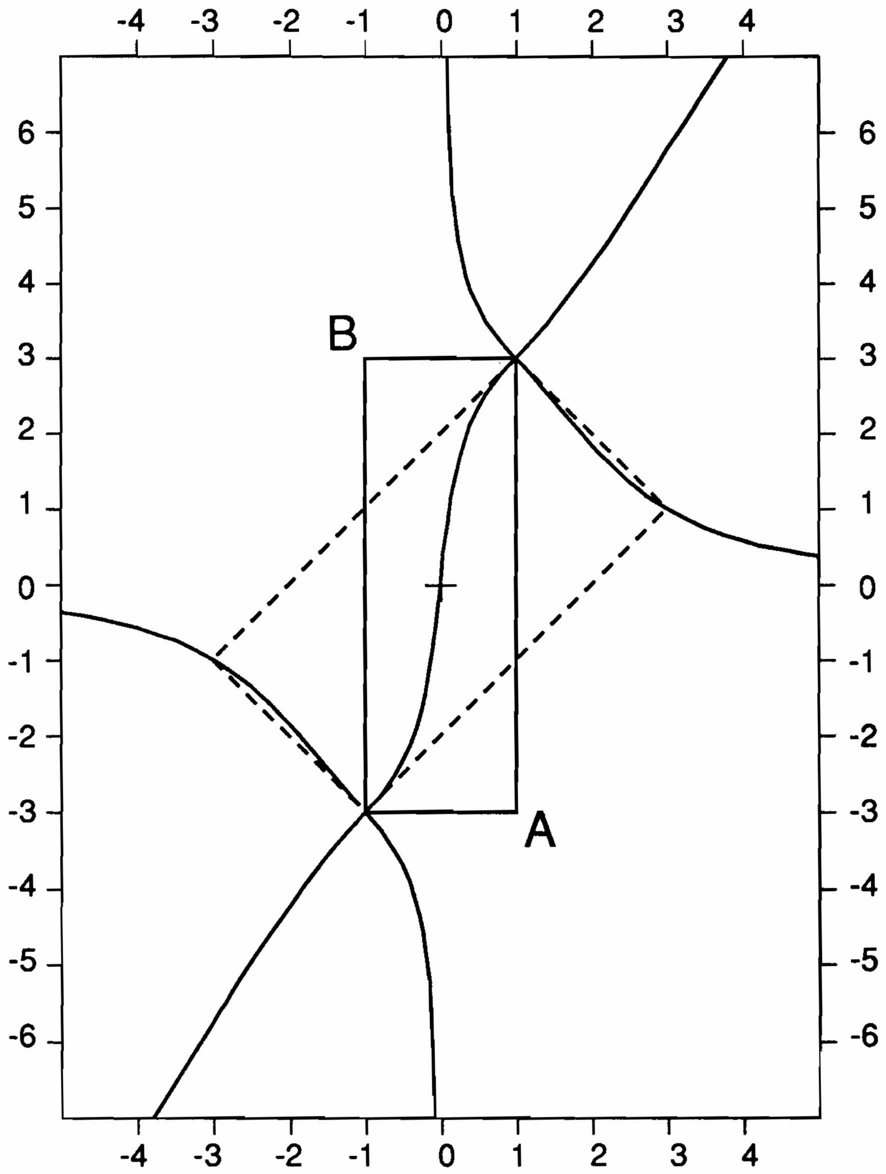


Figure 9A

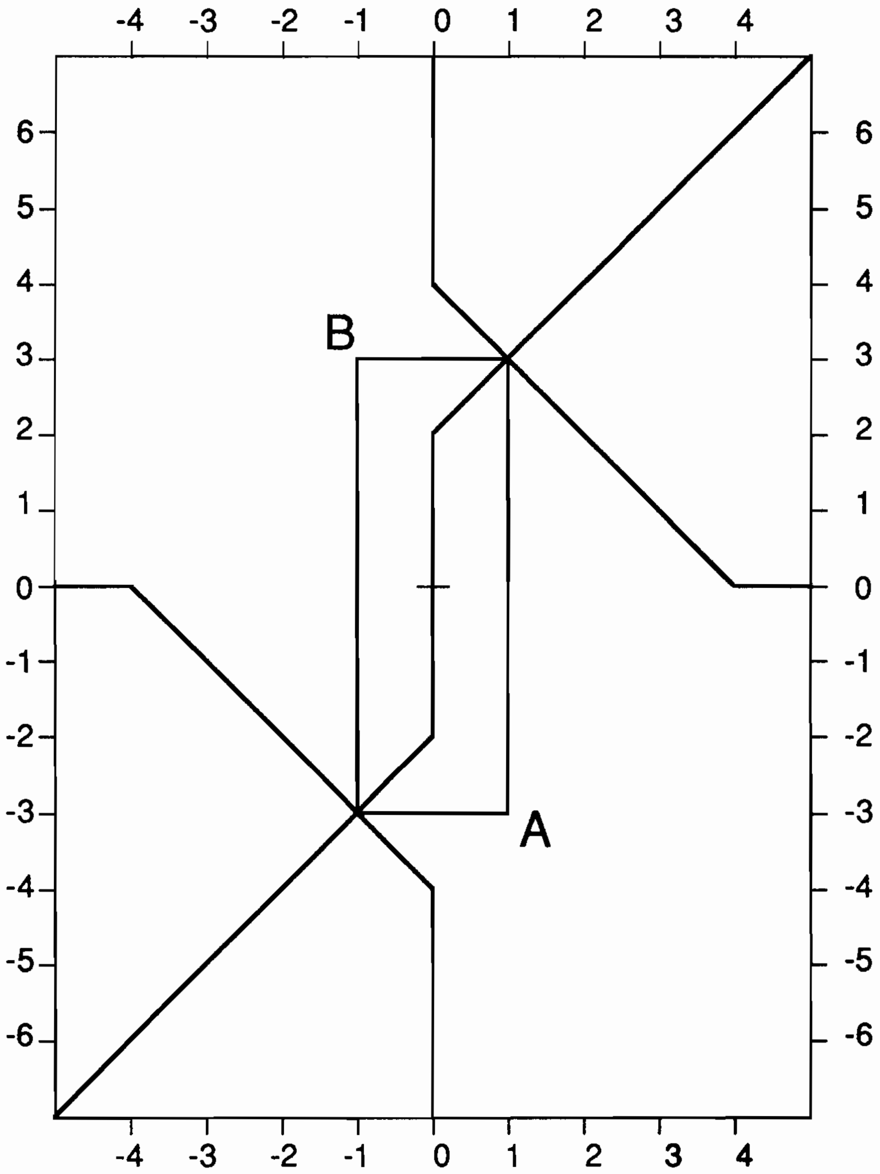


Figure 9B

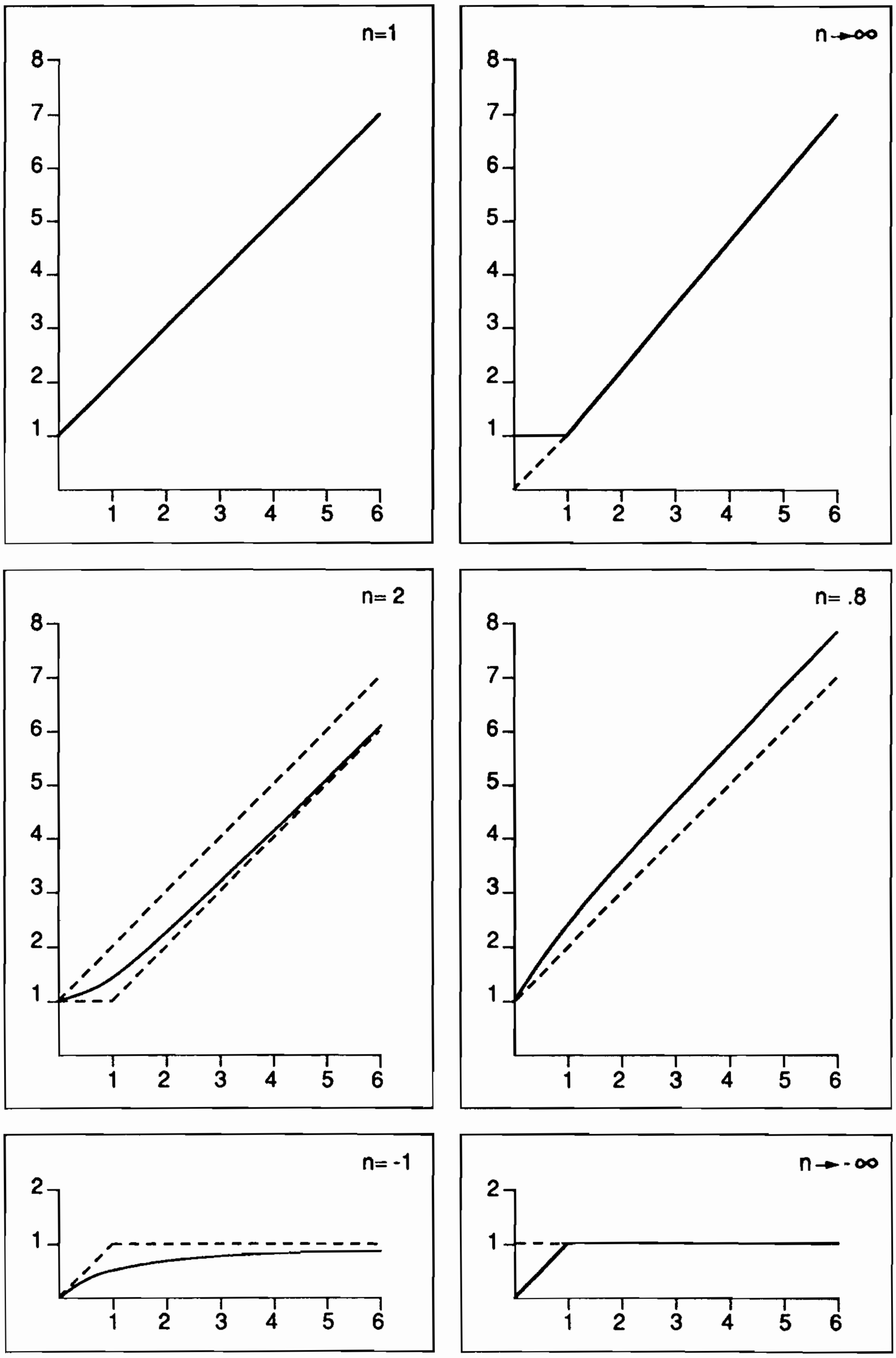


Figure 10

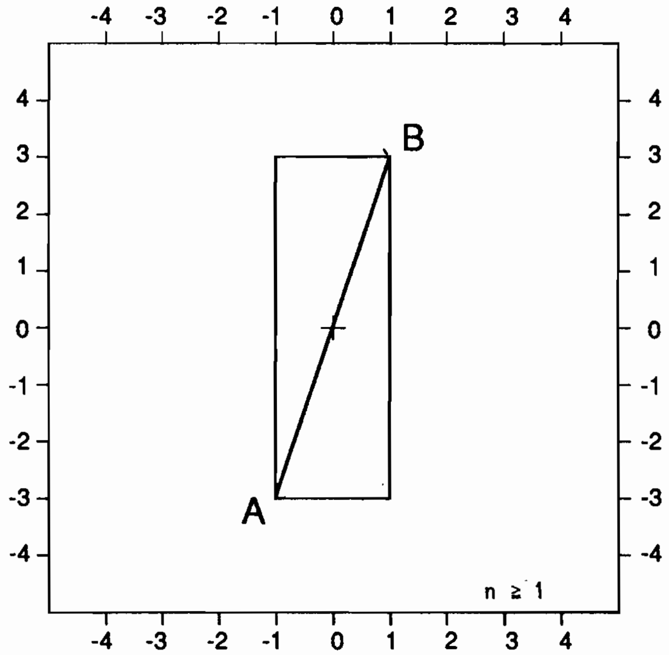


Figure 11A

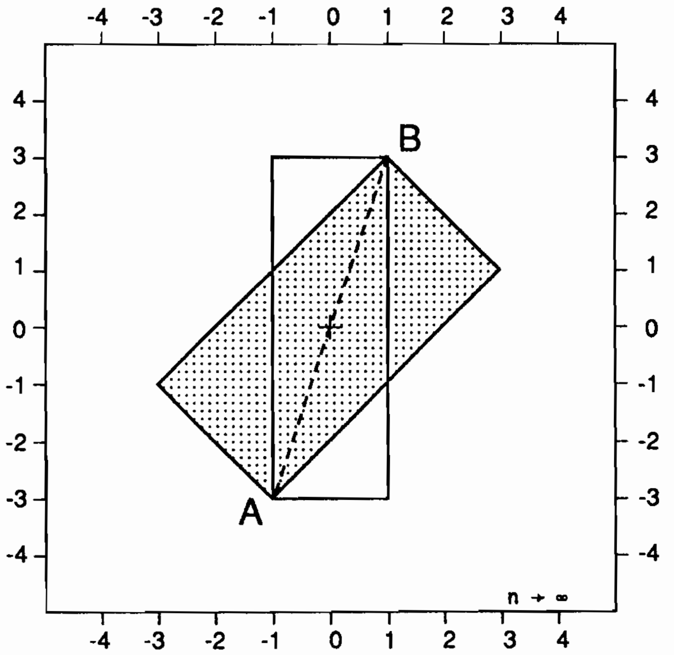


Figure 11B

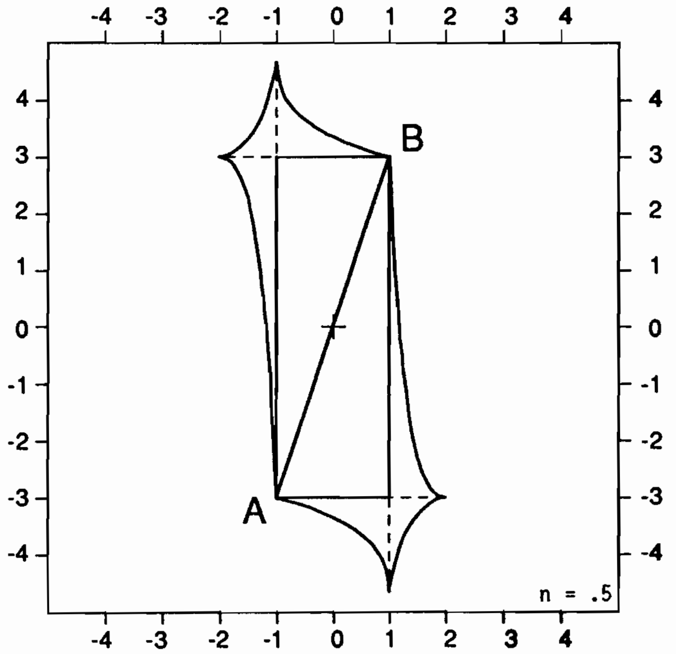


Figure 11C

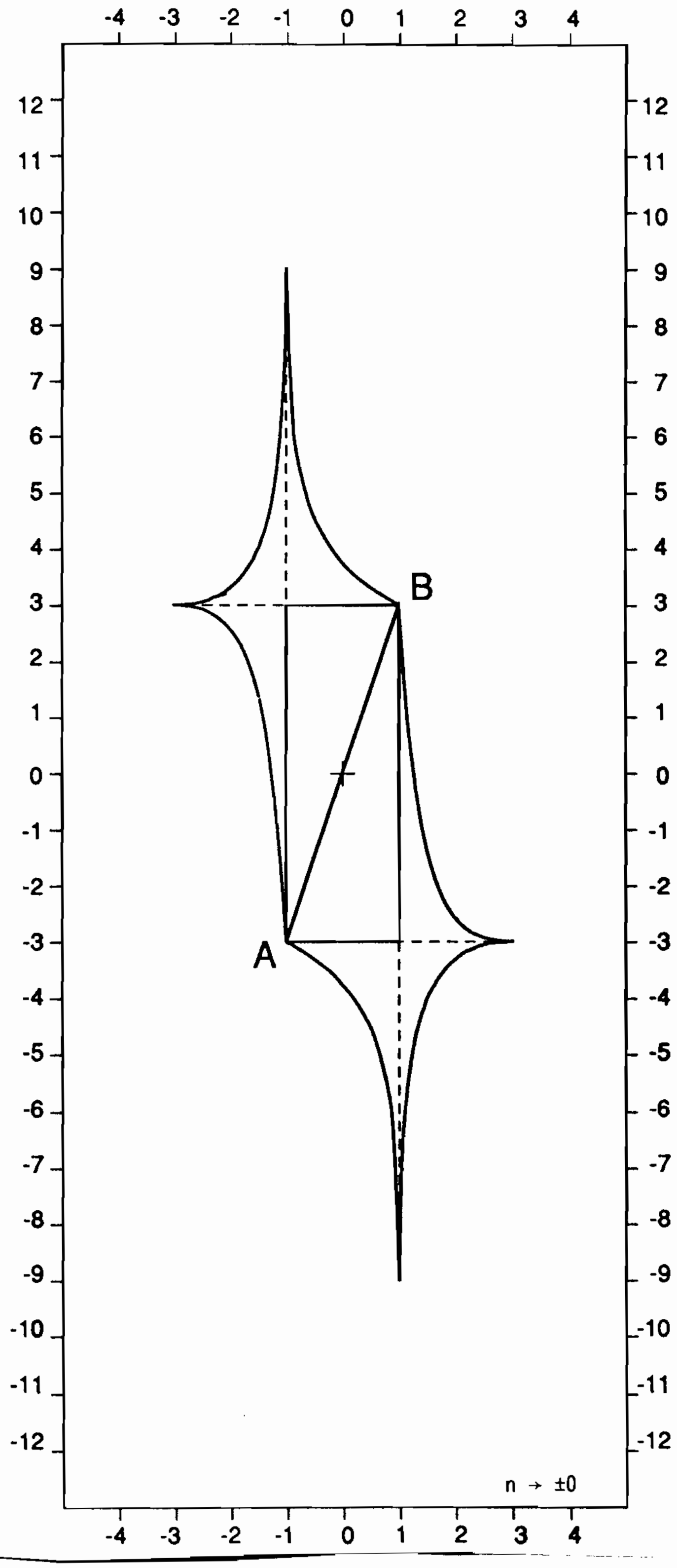


Figure 11D

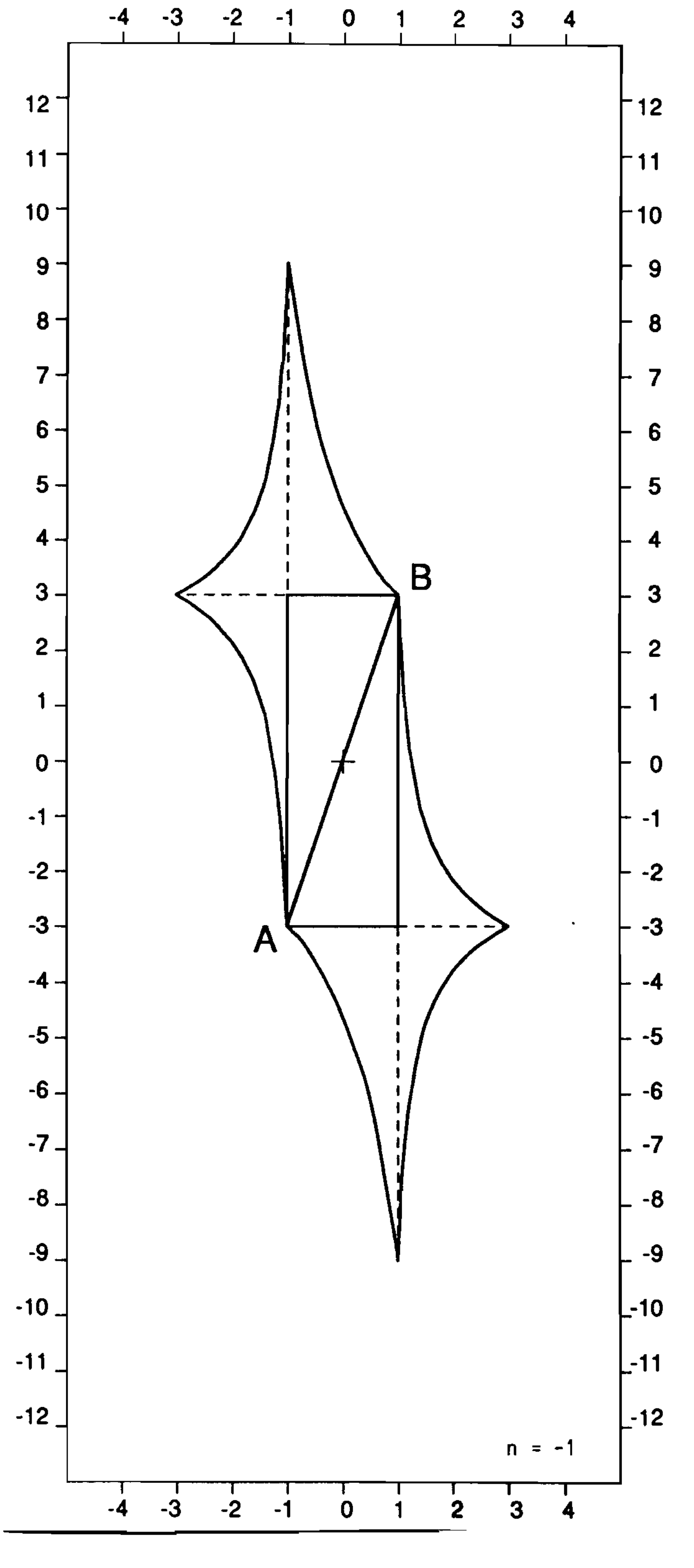


Figure 11E

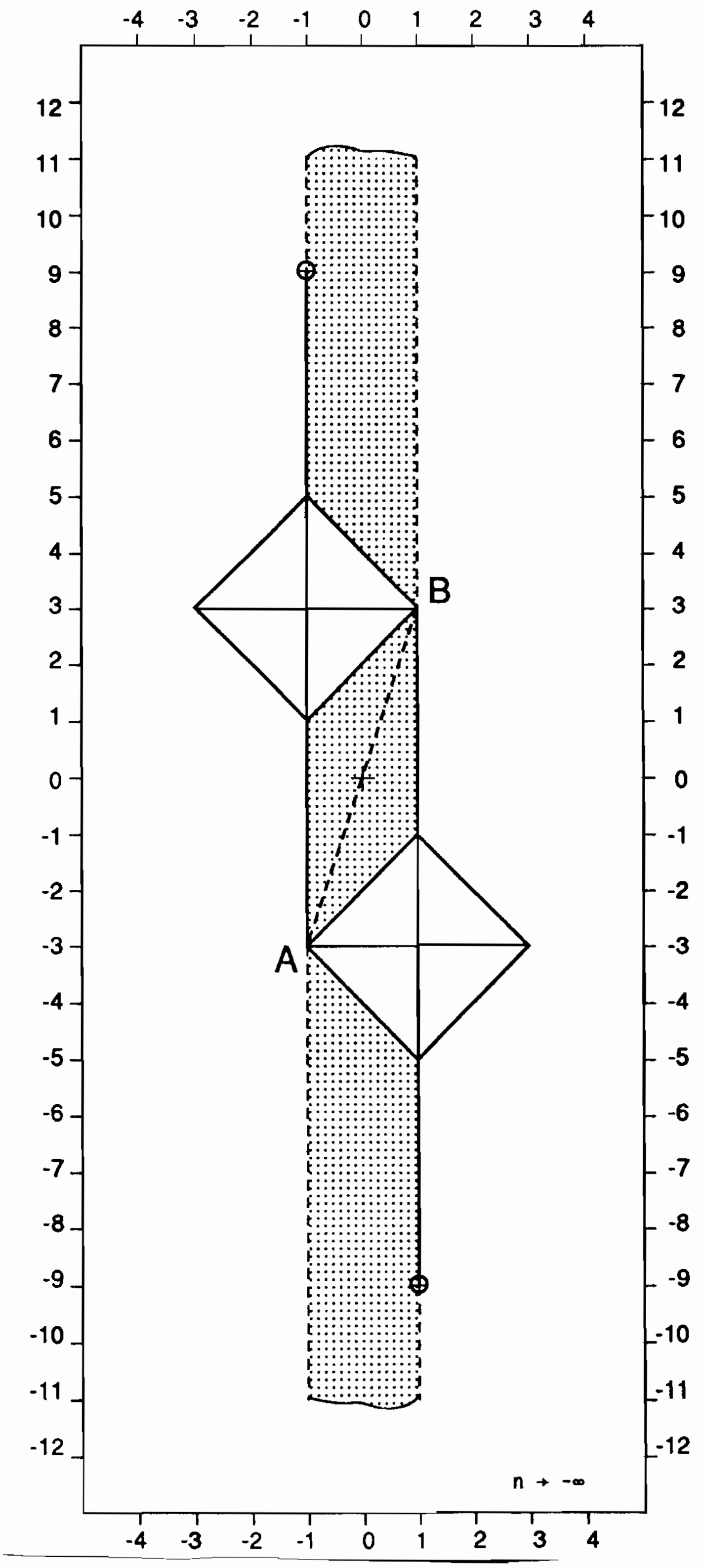


Figure 11F

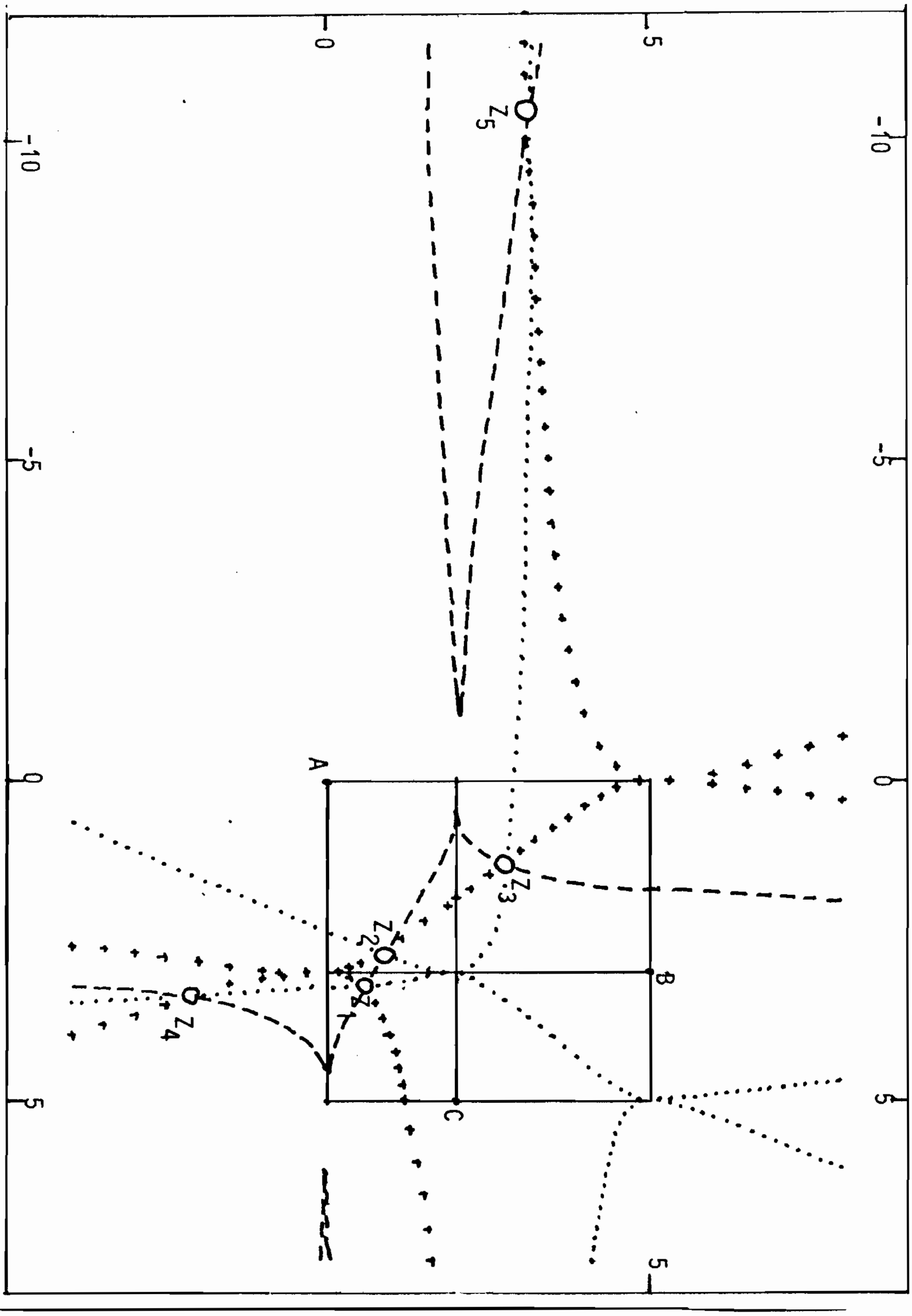


Figure 12

LEGENDS OF FIGURES

Figures 1A - 1F	Equi-distance contours depending upon the value of n.
Figures 1G and 1H	The upper Figure 1G shows the equi-distance contour for $n = -1$. Figure 1H (below) is valid for $n \rightarrow -\infty$. Both figures are cut off, arbitrarily, at $x = \pm 3$ and $y = \pm 3$.
Figure 2	Plot of Minkowski distances d_n (with $n = 2$) against d_m ($m = 1$). Numbers in this graph represent how many points coincide on the same location. These numbers add up to 45. Further explanation is given in the text.
Figure 3	Graph of the 5 points specified in Table 1.
Figure 4	The figure demonstrates that W has larger city-block distance from O than A, whereas V has smaller city-block distance than A. On the other hand, W has smaller Eucidean distance to O than A, whereas V has larger Euclidean distance than A.
Figure 5	The five figures show curves on which points C must be located when it is required that the Minkowski distance between A and C is equal to that between B and C. These curves depend upon the value of the Minkowski parameter n.
Figure 6	The three Figures A to C are based on a Minkowski parameter $n = .5$. Equi-distance contours then become "stars". Figure A shows that a corner of one of those two stars may just touch the curved side of the other star. Figure B shows that for larger value of d, sides of the stars become tangential. Figure C shows that for still larger value of d one of the stars again just touches a curved side of the other star, this time a curved side that is father away than in Figure A.
Figure 7	Location of points at equal Minkowski distance from A and B, when $n = .5$.

Figure 8

Location of points at equal Minkowski distance from A and B when $n \rightarrow 0$. These points are located either on the straight line through the two corners of the rectangle of which A and B are the other two corners. Or they are located on a perfect hyperbola, with branches passing through the two corners of the rectangle.

Figure 9

Location of points at equal Minkowski distance from A and B when $n = -1$ (Figure 9A), and when $n \rightarrow -\infty$ (Figure 9B). In the latter case, the curves become broken straight lines.

Figure 10

Plots of d (ordinate) as a function of y (abscissa), dependent upon the value of n, and for given value $x = 1$.

Figure 11

The figures illustrate the location of points C when it is required that Minkowski distances between A and C, and between C and B, add up to the Minkowski distance between A and B.

Figure 11A illustrates that points C then must be located on the straight line between A and B when $n > 1$. Moreover, when $n = 1$, the additivity is valid for *all* points in the interior of the rectangle of which A and B are two corners, whereas $(-v, w)$ and $(v, -w)$ are the other two corners.

Figure 11B illustrates the situation where $n \rightarrow \infty$. Points C now must be located in the interior of the shaded rectangle, of which A and B are two corners, whereas (w, v) and $(-w, -v)$ are the other two corners.

Figure 11C gives the illustration when $n = .5$. Points C are located on the sides of a curved hexagon.

Figure 11D gives the limiting case when $n \rightarrow 0$. Again we find a curved hexagon, with corners at A and B, but also corners at $\pm(v, -3w)$ and $\pm(3v, -w)$.

Figure 11E gives the illustration for $n = -1$. The curved hexagon looks very similar to that for $n = 0$. In particular, its 6 corners are the same.

Figure 11F shows the result when $n \rightarrow -\infty$. The hexagon is still there, but its sides now are broken straight lines instead of curves.

Moreover, additivity now is valid for all points located in the shaded region, because for all those points the horizontal distance to A or B is smaller than the vertical distance. Minkowski distances between C and A or B then are defined by those horizontal distances, and their sum is equal to the horizontal distance between A and B.

Figure 12 — For the given triangle ABC, the figure shows 5 solutions for points Z such that Z has the same distance (based upon $n = .5$) to each of the three corners of the triangle. Coordinates are specified in Table 3.