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MAXBET solutions for linear relations between k sets of variables

John P. Van de Geer

Department of Data Theory

University of Leiden

ABSTRACT

Given a matrix of correlation between variables that can be partitioned into k sets. Question: can we find weights for the variables in such a way that their weighted sum has stationary value for its variance?
A well-known solution is to take weights from the eigenvectors (eigenvalues then have stationary value). Such a solution prescribes that eigenvectors must have fixed norm. But they ignore the partitioning of the variables, so that it may happen that an eigenvalue solution is "unfair" (is dominated by some of the sets, and ignores other sets).
A possible remedy is to require that weights for each separate set have the same norm (identical sum of squares).
In this monograph, firstly, the problem is simplified (by transforming variables within one set to orthogonal variables). Thereafter, solutions for vectors of weights (for the transformed variables) are discussed. Such vectors are called t, under the condition that $t't = k$. Without further restrictions, this results into eigenvalue solutions. Their disadvantage is that they do not take into account that there is a partitioning into k sets.
However, one may require not only that $t't = k$ but also for each subvector t_i of t, that $t_i' t_i = 1$. Such a solution is called an MB-solution. It implies that partial derivatives of the characteristic equation are equal, but not equal to zero.
If all partial derivatives are equal to zero, singular points are identified, called QSP solutions. The monograph enters into the relation between MB-solutions, and QSP solutions.
This discussion is based on numerical examples. The basis example has k=3, with two variables within each of the three sets. This example is treated in section 13. As a preliminary, examples are analyzed with k=2 in section 7 by taking sets 1 and 2 from the general example, in section 8 by taking sets 1 and 3, and in section 9 by taking sets 2 and 3.).
Intermediate sections discuss special cases and special problems.
Conclusions are, firstly, that there is a special relation between QSP and MB-solutions. Secondly, that such solutions may stand for rather complicated saddle points. Thirdly, that algorithms (to solve for MB or QSP solutions) may encounter tricky difficulties.

CONTENTS

PREFACE

The present monograph is a record of afterthoughts about the problem of linear relations among k sets of variables, discussed earlier in Van de Geer (1864) and (1986). More in particular, the monograph enters into details of what has been called the "MAXBET solution with $t_i' t_i = 1$ ", here abbreviated as MB-solution. In general, solutions among k sets hinge upon a characteristic equation with k parameters. MB-solutions have the property that all k partial derivatives of this equation are equal to each other, but should not be all equal to zero. If all partial derivatives are zero, solutions are found for singular points (or quasi singular points where the charachteristic equation itself differs from zero).

In previous publications I have been rather negligent about the relation between MB-solutions and quasi singular point solutions. The present monograph investigates such relations more closely.

In addition, more attention is paid to the precise nature of MB-solutions (and quasi singular point solutions). It turns out that those solutions may represent rather complicated "saddle points".

More in general: the surface defined by a characteristic equation is a rather tricky one. This implies that algorithms for identifying solutions may encounter unexpected stumbling blocks. Some of these difficulties are indicated.

1 BASIC PROBLEM

1.1 Basic data

Basic data are contained in a matrix X in which columns refer to variables, rows to objects. This matrix is partitioned into k sets: $X = (X_1, X_2,\ldots\ldots X_k)$ and the problem is to find weighted sums X_iv_i in such a way that these k sumvectors are as much as possible "similar". The precise meaning of this "similarity" will be given later.
It will be assumed that columns of X are standardized, so that $R = X'X/n$ is a correlation matrix.

1.2 Simplification

Without loss of generality, each X_i can be replaced by an orthogonal basis.
In particular, let $X_i = P_i\Phi_iQ_i$ where

$P_i'P_i/n = I$

$Q_i'Q_i = I$

Φ_i is a diagonal matrix, with in its diagonal the singular values in descending order. This decomposition of X_i is known as the 'singular value decomposition', abbreviated as SVD. Singular values equal to zero are omitted from the solution. It then follows that $P_i\Phi_i$ is an orthogonal basis of X_i, and that $P_i\Phi_i$has full column rank, equal to the column rank of X_i (which may be less than the number of variables in X_i).
It also follows that some solution X_iv_i is equivalent with a solution $P_i\Phi_it_i$,with $v_i = Q_it_i$.
Moreover, it will be true that $v_i'v_i = t_i'Q_i'Q_it_i = t_i't_i$ so that v_i and t_i have identical norm.

1.3 Notation

Suppose now that X is replaced by a matrix $P\Phi = (P_1\Phi_1, P_2\Phi_2,\ldots\ldots\ldots P_k\Phi_k)$.
We shall use the symbol B for the matrix $B = \Phi P'P\Phi/n$.
Let $P_1\Phi_1$ have m_i columns. Then $P\Phi$ has $\Sigma m_i = m$ columns, and this is also the dimension of B.
B can be partitioned in "blocks". There are diagonal blocks, defined by $B_{ii} = \Phi^2 i$ so that these diagonal blocks are diagonal matrices. And there are off-diagonal blocks, defined by $B_{ij} = \Phi_iP_iP_j\Phi_j/n$.
A diagonal block B_{ii} has trace $\Sigma\Phi^2 i = m_i$. The matrix B–I therefore will have diagonal blocks with trace equal to zero. In the sequel we shall use the notation M = B–I (and $M_{ii} = B_{ii}–I$, whereas $M_{ij} = B_{ij}$ for $i \neq j$).
It will be shown in the sequel that we are interested in solutions for μ_i which can be collected in a diagonal matrix DB in such a way that a diagonal block of DB is equal to μ_iI. In particular: we shall be interested in solutions where B–DB has rank smaller than m.
Such solutions come to the same thing as those for M, and a diagonal matrix DM = DB–I, because B–DB = (B–I) – (DB–I) = M–DM.

1.4 Illustration

Throughout this monograph we shall use an illustration with B as given in Table 1. In this illustration, $k = 3$, and $m_i = 2$. Table 2 gives the corresponding matrix $M = B-I$. Table 3 shows how DB looks like, whereas Table 4 shows how DM looks like, with $x = \mu_1-1$, $y = \mu_2-1$, $z = \mu_3-1$.

2 EIGENVALUE SOLUTIONS

The eigenvalue solution requires $Rv = v\mu$ (with normalization $v'v = k$, say) or, in our simplified version $Bt = t\mu$ (with normalization $t't = k$) which comes to the same thing as $Mt = t(\mu-1)$.
Taking the notation $B_i = (B_{i1}, B_{i2}, ...B_{ik})$ we also can write $B_i t = t_i\mu$.
But this last expression is valid no matter how B is partitioned. In fact, the eigenvalue solution takes no account at all of the partitioning of X.
It is well known that the eigenvalue solution has stationary value for the sum of squared correlations between individual variables x_{ij} and the sum vector Xv – this sum of squares is equal to μ.
We may also say that the eigenvalue solution has stationary value for $v'X'Xv/n = t'Bt = k\mu$.
In anticipation of the sequel it also can be said that the eigenvalue solution requires $(B-DB)t = 0$ where DB is the diagonal matrix μI. It then also follows that $|B-DB| = 0$, where $|B-DB|$ is the determinant of B(B–DB).

3 MAXBET SOLUTION WITH $t_i't_i = 1$

This solution also seeks for stationary value of $v'X'Xv/n = t'Bt$, but now with the additional requirement that $t_i't_i = 1$. It can be shown that such a solution obeys the equation $B_i = t_i\mu_i$ so that $t_i'B_it_i = \mu_i$. This solution does not ignore the way how B is partitioned. We also may write $(B-DB)t = 0$ where DB is now a diagonal matrix with diagonal blocks $\mu_i I$. Again it then follows that $|B-DB| = 0$.

4 GENERALIZATION

In a more general sense, one might require that a solution with $t't = k$ obeys the special conditions $t_i't_i = c_i$ with apriori fixed values of c_i such that $\Sigma c_i = k$.The MB-solution then is the special case where $c_i = 1$. Solutions for t under this more general set of conditions (and such that t'Bt has stationary value) should also obey $(B-DB)t = 0$ with diagonal blocks $\mu_i I$ in the diagonal matrix DB. The values of μ_i which satisfy such a solution are, of course, dependent on the choice of the values c_i.

5 CHARACTERISTIC EQUATION

The reasoning above shows that solutions require $H = |B-DB| = 0$.
In the numerical example, this means that we have to subtract Table 3 from B in Table 1, and we have to develop the determinant. This results into an algebraic equation $H(\mu_1,\mu_2,\mu_3) = 0$, of which the term with highest degree is $\mu_1^2\mu_2^2\mu_3^2$, a term of degree m = 6.
Instead, we may take the equivalent expression $F6 = |M-DM| = 0$ where M and DM are given in Tables 2 and 4. The equation F6 = 0 now has a term of highest degree $x^2y^2z^2$. The advantage of using M instead of B is that F6 does not contain terms of fifth degree (of degree (m–1)).
In fact, we obtain

$$\begin{aligned} F6 = x^2y^2z^2 & - .16x^2y^2 - .04x^2z^2 - .64y^2z^2 - .51x^2yz - .63xy^2z - .70xyz^2 \\ & - .004x^2y - .062x^2z - .296y^2z - .212xy^2 - .060xz^2 - .480yz^2 \\ & - .780xyz \\ & + .0049x^2 - .0231y^2 - .0220z^2 + .0230xy - .0400xz - .0683yz \\ & + .00474x + .00538y - .00010z + .000367 = 0. \end{aligned}$$

If we require x = y = z, the equation above results into $x^6 - 2.68x^4 - 1.822x^3 - .1255x^2 +.01032x - .000367 = 0$ and the roots of this equation are equal to the eigenvectors of B, with 1 subtracted. These solutions for $(\mu - 1)$ are given in Table 5.

6 PARTIAL DERIVATIVES OF THE CHARACTERISTIC EQUATION

Take again the generalized formulation: we want stationary value for t'Bt, conditional upon the requirements $t_i't_i = c_i$ (c_i is a fixed constant) with $\Sigma c_i = \Sigma t_i't_i = t't = k$.
We then have $B_{i.}t - t_i\mu_i$ so that $t_i'B_{i.}t = t_i't_i\mu_i = c_i\mu_i$ and if follows that $t'Bt = \Sigma c_i\mu_i$.
But the latter formulation of the problem is equivalent with a solution $H = |B - DB| = 0$. It therefore must be valid that $\delta H/\delta\mu_i \propto c_i = t_i't_i$.
The same then must be valid for the partial derivatives of t'Mt in relation with those of $F = |M-DM|$: partial derivatives of F with respect to (μ_i-1) will be proportional to the value of $t_i't_i$. After all, the change from B to M is nothing but a rigid translation : from the origin $(\mu_i = 0)$ to an origin $(\mu_i = 1)$.

7 FIRST EXAMPLE; $k = 2, m_1 = m_2 = 2$

7.1 Characteristic equation

For a first illustration we take only the first two sets of the example given in Table 1 (for B) and Table 2 (for M). On the basis of M, the characteristic equation becomes:
$F12 = x^2y^2 - .04x^2 - .070xy - .64y^2 - 06x - .48y - .022 = 0.$
(This equation could also be derived from the full equation F6 = 0 given in section 5: take from this equation only the terms which include z^2 and then set z^2 equal to 1.)

7.2 Graph

Figure 1 gives a graph of F12 = 0. Typically, such a graph consists of four "branches", delimited by asymptotes at $x = \pm.8$ and $y = \pm.2$ (the diagonal values in M). Eigenvalue solutions are found at the points of the graph where x = y (or: where the branches are intersected by the line x = y). Numerical values are shown in Table 6.

7.3 Partial derivatives

$G_x = \delta F12/\delta x = 2xy^2 - .08x - .70y -.06$
$G_y = \delta F12/\delta y = 2x^2y - .70x - 1.28y - .48$

The derivative of x with respect to y is then found as $dx/dy = -G_y/G_x = -(t_2't_2)/(t_1't_1)$.
The value of $-dx/dy$ indicates the slope of the tangential at a point of the curve. It thus can be seen that this slope always is negative. The figure shows that the first eigenvalue solution is dominated by the first set ($t_1't_1$ is relatively large compared with $t_2't_2$ – the tangential has a flat slope – see also Table 6). Similarly: the second eigenvalue solution is dominated by the second set. The third solution is rather neutral, and the fourth is again dominated by the first set.

7.4 Relations between G_i and $t_i't_i$

The present paragraph is a short interlude. One may wonder what precisely is the relation between the value of G_i and that of $t_i't_i$? Let us take the generalized case for two sets, with $t_i't_i = c_1$ and $t_2't_2 = c_2$.
The proportionality relation shown earlier, implies that $G_x = c_1/\gamma$ and $G_y = c_2/\gamma$, where γ is some proportionality constant.
It follows that

$c_1x = \gamma x G_x$
$c_2y = \gamma y G_y$

so that $c_1 x+c_2y = \gamma (xG_x + yG_y)$ which identifies γ without making use of the values of t_1 or t_2. Moreover, $c_1x + c_2y = t'MDt = t'Mt$ (where $F12 = |M-DM|$) and in the present illustration it can be shown that $xG_x + yG_y = 2x^2y^2 + .06x + .484y + .044$.

The reader may verify that for the first eigenvalue of the present example, where $x = y = 1.3399$, we obtain

$$c_1x + c_2y = (c_1 + c_2)x = 2x = 2.6798$$
$$xG_x + yG_y = 2x^4 + .54x + .044 = 7.2136$$

and $(1/\gamma) = (7.2136)/(2.6798) = 2.6919$.

7.5 MAXBET solutions

7.5.1 MAXBET solutions with $t_1't_1' = t_2't_2 = 1$ will in the sequel be called MB-solutions, or, shortly, MB1, MB2, etc.
Such solutions imply that $G_x = G_y$. Therefore in the graph of Figure 1 such solutions can be identified as points on the branches where the tangential has slope –1. Figure 1 gives the illustration. Numerical values are given in Table 7.

7.5.2 Define $G_d = G_x - G_y$. The numerical solution for $G_x = G_y$ becomes: $G_d = xy^2 - x^2y + .31x + .29_y + .21 = 0$.
The graph of this equation of third degree will have three branches, with asymptotes at $x = 0$, $y = 0$, and $x = y$. This graph is shown in Figure 2. Clearly, MB-solutions are the points where the branches of G_d intersect those of F12.

7.5.3 Given the location of the asymptotes there is necessarily an intersection between the first branch of F12 and the Northeastern branch of G_d. Also, the SE branch of G_d must necessarily intersect branches 2 and 3 of F12. Finally, the SW branch of G_d must intersect branch 4 of F12. On the other hand, intersections of the SW branch of G_d and branch 3 of F12 may occur, but they are not necessarily there.
In the example they do occur, and this seems related to the finding that branch 3 has two bending points, so that there also are three MB-solutions on branch 3. How should this be explained?

7.6 F12 as a function of x and y

7.6.1 Let F12 be a function of x and y, so that x and y may take any value, resulting into some value of F12, not necessarily equal to zero. This implies that F12 now defines a surface in a three-dimensional space with F12, x and y as coordinates axes. The graph of $F12 = 0$ (Figure 1 or Figure 2) shows the intersection of this surface with the plane $F12 = 0$.

7.6.2 To obtain a better understanding of the surface it is helpful to see how it intersects with planes for constant value of x. We may write $F12 = A_yy^2 + B_yy + C_y$ where A_y, B_y, and C_y are functions of x. For fixed x, therefore, F12 becomes the equation of a parabola. The apex of the parabola is located at $y_a = -B_y/2A_y$ because for the apex it must be true that $G_y = 0$. Moreover, this point is midway between the two solutions for y from $F12 = 0$ (given fixed x).

7.6.3 We want to inspect what happens with the apex when the fixed value of x is gradually shifted from ∞ to $-\infty$.

(i)
- For $x \rightarrow \infty$, we find that $y_a \rightarrow 0$, while $F12 \rightarrow -\infty$.
- For x >.8, the apex lies midway between branches 1 and 2, and its value of F12 is negative.
- For x =.8, the value of A_y becomes zero, and the parabola degenerates into a straigt line
  $F12 = B_y y + C_y$.
- At some value of x >.8, the value of F12 at the apex must have a maximum: the point where the minimum of the parabola attains a maximum. In other words, we have a maximin 1 for F12, a saddle-point.
- But this then also must be true for the parabola obtained when y is fixed at this value for y_a. The latter implies that at the maximin not only $G_y = 0$, but also $G_x = 0$. It follows that the maximin is midway between branches 1 and 2, both the horizontal x-direction and in the vertical y-direction.
- The maximin point therefore is also located on the NE-branch of G_d (see Figure 2).

(ii)
- For .8 > x > –.8, the apex becomes the top of a parabola, with F12 larger than zero.
- Again, somewhere in this interval the value of F12 for the apex must have a minimum. At such a point we have a minimax, with $G_x = G_y = 0$. The point will be located midway between branches 2 and 3, both in the vertical and in the horizontal direction.

(iii)
- For x = –.8 the parabola degenerates again to a straight line.
- For x < –.8, the apex becomes a minimum again, with F12 negative. At some point we find a maximum: the point midway between branches 3 and 4.

7.6.4 The same story as above can be told for the parabola's obtained when y is fixed. We now will find the maximin between branches 1 and 2 at some value of y > .2, the minimax between branches 2 and 3 at some value –.2 < y < .2, and the maximin between branches 3 and 4 at some value at some value y < –.2.

7.7 Quasi singular points

7.7.1 Section 7.6 defined maximin and minimax points, where $G_x = G_y = 0$. Such solutions are independent of the value of F12. They will be called solutions for quasi singular points QSP. If for such solutions it would be true that F12 = 0, they represent "proper" singular points.

7.7.2 The equality $G_x = G_y = 0$ can be transformed into an equation of fifth degree in x (or in y). In principle, therefore, there could be five QSP solutions.
The latter is illustrated in the hypothetical example of Figure 3 (we come back to this example in section 13.8). In this graph both branches 2 and 3 have bending points, and between them

are 3 QSP solutions: two where the two branches come close together, and a third solution between the first two, where the branches are again relatively far apart.

7.7.3 In the present example there are only three QSP solutions (the other two are imaginary points). They are listed in Table 8, and graphs are shown in Figure 4.

(i) At F12 = .033233, branches 3 and 4 intersect. This explains why for F12 = 0 the third branch obtains bending points. In figure 4C the third branch is defined by the oblique M-shape, and the fourth by the sharp-angled L-shape curve left-below. Clearly, the concept of "branch" becomes artificial: it wouldbe much more natural to say that in Figue 4C there are two intersecting smooth curves, instead of a rotated M touching an inverse L.

(ii) A t F12 = .000325 branches 2 and 3 intersect. At F12 = 0 there is a small gap between these two branches, and MB4 and MB5 emerge, very close to the middle QSP point. But at this QSP point itself, there is no MB-solution (for the given value F12 = .000325), because at the point of intersection none of the two tangentials has slope –1.

(iii) At F12 = .549331, branches 1 and 2 intersect. At the same time branches 3 and 4 "degenerate". This point will be more fully discussed in section 13.8.

(iv) If between branches 1 and 2 there would be a QSP solution with value of F12 very close to zero, one should expect bending-points in branch 2.

(v) In general, one should not expect to find bending-points in branches 1 and 4. There cannot be a QSP solution to the NE of branch 1, or to the SW of branch 4.

(vi) In the graphs for the QSP solutions, the asymptotes remain identical, because the value of A_x or A_y is not affected by the choice of the value of F12.

(vi) Just for the record: in the present example branch 3 has the following two bending points:
(x y) = (–.325 –.444) with tangential slope –1.495,
(x,y) = (–.663 –.090), with tangential slope –.446

We shall come back to these bending points in the appendix.

7.8 Orthogonality

7.8.1 Take the 4 × 4 matrix formed by the first four columns and rows of M in Table 2. Let this matrix be called M_4.
Suppose v_a and v_b are two eigenvectors of M_4, normalized to v'v = k=2. Their corresponding eigenvalues are λ_a and λ_b. Then it will be true that $v_a'M_4v_b = v_a'v_b\lambda_b$ and at the same time that $v_b'M_4v_a = v_b'v_a\lambda_a$.
It follows that, if $\lambda_a = \lambda_b$, it must be true that $v_a'v_b = 0$: eigenvectors are orthogonal.

7.8.2 On the other hand, take two solutions t_a and t_b for MAXBET.
I.e., $t_{1a}'t_{1a} = 1$, $t_{2a}'t_{2a} = 1$, $t_{1b}'t_{1b} = 1$, and $t_{2b}'t_{2b} = 1$.
It then follows that $t_a'Mt_{a4}'t_b = t_{1a}'t_{1b}x_a + t_{2a}'t_{2b}y_a = t_{1a}'t_{1b}x_b + t_{2a}'t_{2b}y_b$.
E.g. take the solutions MB1 and MB2, where $t_{11}'t_{21} = -.943$, and $t_{12}'t_{22} = .926$.
For MB1 we have $x_1 = 1.599$ and $y_1 = .955$. For MB2 we have $x_2 = -.014$ and $y_2 = -.687$.

The property above implies that

$(-.943)(1.599) + (.926)(.955) = (-.943)(-.014) + (.926)(-.687)$.

I must confess that the geometrical implications of such equalities are not clear to me.

7.9 Idealisation

The present example can be approximated by an "ideal" case where M12 is a diagonal matrix:

$$M12 = \begin{pmatrix} .7688 & \\ & .3302 \end{pmatrix}$$

The characteristic equation now becomes: $x^2y^2 - .04x^2 - .70xy - .64y^2 - .0964x - .3857y - .022$ which differs from F12 only in the coefficients of x and y. The equation above can be factorized as:

$$\{(x-.8)(y-.2) - .5910\}\ \{(x+.8)(y+.2) - .1090\} = 0$$

and the graph of this equation shows two perfect hyperbola's, with the same asymptotes as the graph of F12. There are four MB-solutions, in two pairs which are symmetric with respect to the two points where asymptotes intersect: (.8 .2) and (–.8 –.2). They are listed in Table 9, together with their corresponding solutions for t. The latter, of course, become very simple, in this "diagonalized" construction. The first of these artificial MB-solutions is quite close to MB1, the second to MB2, the fourth to MB6. But the third solution is not so close to MB4 as one might have wished.

8 SECOND EXAMPLE WITH $k = 2, m_i = 2$.

For another example we take relations between first and third set of Table 2. The characteristic equation now becomes

$$F13 = x^2z^2 - .16x^2 - .63xz - .64z^2 - .212x - .296z - .0231 = 0$$

A graph is shown in Figure 5. This graph shows that there are bending points in the second and in the third branch. Numerical values for eigenvalue solutions and MB-solutions are given in Table 10. There are three MB-solutions on the second branch.

9 THIRD EXAMPLE WITH $k = 2$, $m_i = 2$

We now take relations between second and third set of the matrix in Table 2. We have

$$F23 = y^2z^2 - .16y^2 - .51yz - .04z^2 - .004y - .062z + .0049 = 0.$$

Figure 6 gives a graph of F23. There do not seem to be bending points in branches 2 or 3. Eigenvalues, and the four MB-solutions are specified in Table 10.

10 HOW TO IDENTIFY MB-SOLUTIONS ?

10.1 Graphically

In the situation with $k = 2$ the MB-solutions can be identified from a graph of $F12 = 0$, although with limited precision. Such graphical solutions are, at least, good initial estimates for a more precise numerical algorithm. But with k larger than 2, such a graphical method becomes cumbersome if not unfeasible.

10.2 On lines with slope 1 through intersections of asymptotes

The discussion in section 7.9 suggests that MB-solutions might be found close to a line with slope 1 through an intersection point of the asymptotes. For the example in section 7, this hunch bears out quite well. Take the line $(x-8) = (y - .2)$, or $x = -y+6$. Equation F12 can be re-written as $y^4 + 1.2y^3 - .1.02y^2 - 1.008y - .0724 = 0$.
Two solutions are (x y) = (1.577 .977), close to MB1, and (–.051 –.651) close to MB2. Also, take the line $(x+.8) = (y+.2)$, or $x = y-.6$. Equation F12 becomes $y^4 - 1.2y^3 - 1.02y^2 - .072y - .0004 = 0$.
This gives a solution (–1.042 – .514), close to MB6, another solution (–.606 –.006) close to MB4, and a third solution (–.671 –.071) close to MB5. But MB3 cannot be approximated in this way.
Section 7.8 shows that points on $F12 = 0$ located on a line with slope 1 have solutions for t which are orthogonal. In fact, we find $t_1't_2 = -.0617$, almost orthogonal. But other inner products are more different from zero : $t_4't_5 = -.404$, $t_4't_6 = .341$, and $t_5't_6 = .238$.

10.3 Alternating algorithms

We have tried some alternating algorithms. One of them says: for some selected initial value of y calculate x from $F12 = 0$, and for this value of x calculate a new y from $G_d = G_x - G_y = 0$. Repeat until convergence. This algorithm may be called FxGy. Its companion is FyGx: calculate y from $F12 = 0$, and x from $G_d = 0$. These two algorithms are rather slow, even with good initial guesses. FyGx appears to converge towards MB1, MB2, MB6, MB4, MB5, whereas FxGy converses towards MB3 or becomes oscillating.

We also tried an algorithm that can be easiest explained in geometrical terms. Start with some estimate of y, and calculate x from F12 = 0. Then calculate a point on G_d, in such a way that the two points are located on a straight line with slope –1. This gives an updated estimate of y. Repeat until convergence. There are obvious variations of this principle, by interchanging the order or x and y, or of F12 and G_d. Given a good initial guess, such algorithms work very fast if they converge. But they may go astray by letting x or y grow towards infinity, or may go astray by being trapped in an oscillation between points on different branches of F12 = 0. Actually, the problem is quite straightforward, from the point of view of numerical analysis: find a solution for F12 = 0 that satisfies $G_d = 0$ (or vice versa). My approaches towards solving the problem are quite amateurish. But they may give a hint to experts who are more professional in these matters.

11 PERFECT EXAMPLE WITH $k = 2, m_1 = m_2 = 3$.

11.1. A permutation of a diagonal matrix.

The present example is somewhat superfluous, because it is nothing but a generalization of Section 7.9. It may, nevertheless, illustrate some points more clearly. The example is "perfect" in the sense that M_{12} is a permutated diagonal matrix. See Table 12. The characteristic equation becomes the product of three factors:

$(x- 8)(y+.2) = .09$

$(x+.6)(y-.3) = .01$

$(x+.2)(y+.1) = .04$

Each factor defines a hyperbola, as illustrated in Figure 7. Eigenvalues are found where these hyperbola's are intersected by a line x = y. The numerical values are: (.833 – .283) for the first hyperbola, (.311 – .611) for the second, and (.056 – .356) for the third.

The figure shows that the hyperbola's have intersections, at six points. There are six branches in the figure (two branches for each of the three hyperbola's) and it is quite obvious that it is very artificial to define branches in terms of the order in which they are intersected by a line with slope 1. There are 6 MB-solutions, listed in Table 13, together with their (trivial) solutions for t.

11.2 Singular points where hyperbola's intersect

Where hyperbola's intersect, we have a "singular point". At such points it will be true that

$\partial F/\partial x = G_x = 0$

$\partial F/\partial y = G_y = 0$

Nevertheless, at such points we do not have an MB-solution, in spite of the fact that G_x and G_y are equal. More precisely: at those singular points we find two tangentials. Their slope cannot be defined in the usual way, because the expression $dx/dy = -G_y/G_x = 0/0$ is meaningless.

11.3 Slope of tangentials at singular points

The slope of tangentials at singular points can be identified as follows.

Define $G_{xx} = \partial G_x/\partial x$

$G_{xy} = G_{yx} = \partial G_x/\partial y = \partial G_y/\partial x$

$G_{yy} = \partial G_y/\partial y$

Assuming that these second order partial derivatives of F are not all equal to zero, slope c of a tangential at a singular point can be found from $G_{yy} + 2G_{xy}c + G_{xx}c^2 = 0$.

An MB-solution could be possible, if there is a solution $c = -1$, which would imply $G_{xy} = (G_{xx}+G_{yy})/2$, with a second solution for c equal to $-G_{yy}/G_{xx}$. Or: both solutions for c are equal to -1, which implies $G_{xx} = G_{xy} = G_{yy}$.

12 PERFECT EXAMPLE WITH k = 3

12.1 Characteristic equation

Suppose that with k = 3 all off-diagonal matrices B_{ij} are diagonal. This implies that by re-arranging rows and columns, B can be changed into a matrix in which all off-diagonal blocks are zero matrices. A diagonal block of the re-arranged matrix will have diagonal elements (ϕ_{1j} ϕ_{2j} ϕ_{3j}) (j = 1,2,3).

Define

$(\phi_{1j}-\mu_{1j}) = -x$

$(\phi_{2j}-\mu_{2j}) = -y$

$(\phi_{3j}-\mu_{3j}) = -z$

A diagonal block of the re-arranged matrix then will have the form

$$H = \begin{pmatrix} -x & p & q \\ p & -y & r \\ q & r & -z \end{pmatrix} \text{ with}$$

$$|H| = -xyz + r^2x + q^2y + p^2z + 2pqr = 0.$$

12.2 Factors of characteristic equation

|H| is one of the three factors of equation F = |M|. The surface described by |H| = 0 has its special properties.

(i) Suppose z = 0. Then |H| = 0 describes a straight line: $r^2x + q^2 + 2pqr = 0$. Similarly for y = 0, and x = 0.

(ii) These three straight lines are sides of a triangle, with angular points at:

$(-2pq/r \quad 0 \quad 0)$,

$(0 \quad -2pr/q \quad 0)$, and

$(0 \quad 0 \quad -2qr/p)$.

(iii) The surface includes the point:

$(-pq/r \quad -pr/q \quad -qr/p)$

Moreover, the three straight lines through the latter point , and perpendicular to the planes $x = 0$, $y = 0$, and $z = 0$, respectively, are located on the surface. The surface described by I $|H| = 0$ therefore does not fall apart into separate "branches". Instead, any point on the surface can be reached from any other point on the surface, along a path that remains on the surface.

12.3 Trivial solutions for t.

Since the example is "perfect" in the sense that no further rotations of the three sets will be needed, it follows that MB-solutions for t_i will be vectors in which all elements are zero, except one element equal to ± 1. This implies that H must be accompanied by vectors w, say, of which all elements are equal to ± 1, with $Hw = 0$. It is then easy to solve for the corresponding values of x, y and z. E.g. with $w' = (1 \ 1 \ 1)$, it follows immediately that $Hw = 0$ implies $(x \ y \ z) = (p+q \ \ p+r \ \ q+r)$, For this solution of $(x \ y \ z)$, the matrix H obtains rank 2. Moreover, for these solutions

$\partial|H|/\partial x = \partial|H|/\partial y = \partial|H|\partial z$ so that we in fact have MB-solutions.

Table 14 gives the four possible solutions for w, with their corresponding solutions for (x y z) and for the partial derivatives.

12.4 Haywood case

When we take any of the four solutions for (x y z) given in Table 14, the matrix H becomes a matrix of rank 2. But suppose we take the solution $(x \ y \ z) = (-pq/r \ -pr/q \ -qr/p)$ which is the double point indicated in section 12.2. (iii).

H now becomes a matrix

$$J = \begin{pmatrix} pq/r & p & q \\ p & pr/q & r \\ q & r & qr/p \end{pmatrix}$$

which is a matrix of rank 1. For this choice of (x,y,z), all partial derivatives of $|H|$ will be equal to zero. There is no unique solution for w. All rows of J are proportional to the vector (pq pr qr), and any vector w which is orthogonal to the latter vector, will result into $Jw = 0$. This solution for J presents us with a so-called "Haywood case", usually discussed for the situation that H is a correlation matrix.

In general, matrix J will not be associated with an MB-solution.

13 EXAMPLE WITH $k = 3, m_i = 2$.

13.1 Characteristic equation and its partial derivatives

For a last example we use the data in Table 2. The characteristic equation for M in Table 2 has been given before as F6 = 0 (section 5). The six eigenvalue solutions were listed in Table 5. Given F6, partial derivatives are

$G_x = \partial F6/\partial x =$

$2xy^2z^2$	$-.32xy^2$	$-.08xz^2$	$-1.02xyz$	$-.63y^2z$	
$-.70yz^2$	$-.008xy$	$-.124xz$	$-.212y^2$	$-.060z^2$	$-.708yz$
$+.0098x$	$+.0230y$	$-.0400z$	$+.00474$		

$G_y = \partial F6/\partial y =$

$2x^2yz^2$	$-.32x^2y$	$-1.28yz^2$	$-.51x^2z$	$-1.26xyz$	
$-.70xz^2$	$-.004x^2$	$-.592yz$	$-.424xy$	$-.480z^2$	$-.708xz$
$-.0426y$	$+.0230x$	$-.0683z$	$+.00538$		

$G_z = \partial F6/\partial z =$

$2x^2y^2z$	$-.08x^2z$	$-1.28y^2z$	$-.51x^2y$	$-.63xy^2$
$-1.40xyz$	$-.062x^2$	$-.296y^2$	$-.120xz$	$-.960yz$ $-.708xy$
$-.0440z$	$-.0400x$	$-.0683y$	$-.00010$	

Table 15 again lists the eigenvalues of M. But the table now also gives the corresponding eigenvectors t (normalized to $t't = k$), the values of $t_i't_i$ (with sum 3), and the values of G_x, G_y, and G_z (proportional to those of $t_i't_i$).

Clearly, the example confirms that sets are treated "unfairly" in the eigenvalue solutions. The first one is relatively dominated by the first set. The second more or less ignores the first set, and so does the fifth. The sixth solution again depends almost entirely on the first set alone. Third and fourth solution are more or less "fair" (differences between $t_i't_i$ are not very large). However, for these two solutions the eigenvalue itself is almost equal to $\mu = 1$(or $\mu - 1 = 0$). Such eigenvalues close to unity should be expected in a random correlation matrix and therefore are not very revealing about relations between sets. The example therefore also illustrates why we may be interested in solutions where $t_i't_i = 1$. *)

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*) A related problem appears in multiple regression (Van de Geer 1983, 1986). In simple multiple regression we have two sets: the predictor set with m_1 variables, and the predicand with single variable ($m_2 = 1$) "Good" prediction requires a solution where X_1t_1 is almost equal to X_2t_2. A solution for t can be approximated by taking the eigenvector of the correspondence matrix B with lowest eigenvalue. Nevertheless, such a solution my fail, because it may depend mainly on interdependence between variables in the predictor set. Such a solution implies that t_2 may be close to zero.
The multiple regession solution requires that t'Bt is minimized given that $t_2 = 1$, no limit on t_1. This solution also may be disappointing, because $t_1't_1$ may be very large (which would imply that small changes in the predictor variables can have dramatic effects on the prediction).
In such cases, a "ridge regression" solution has been forwarded. One such ridge regression solution would entail $t_2 = 1$, and $t_1't_1 = 1$: the MB solution wil be a ridge regression solution.

13.2 Branches

F6 = 0 defines a three-dimensional surface. Graphs can be made of its intersection with a plane for constant value of x, of y, or of z. In general, such intersections will show four branches – we shall come back to this topic in the next sections. Such intersections do not clarify that F6 = 0 actually has six branches. We therefore have drawn Figure 8 which shows the intersection of the surface F6 = 0 with the plane defined by y = z. In this figure the six eigenvalues of M are found at the points where x = (y = z). Clearly, by taking a fixed value of y = z, the equation F6 = 0 allows for four solutions of x, whereas for a fixed value of x there will be two solutions for y = z. This is demonstrated in Figure 8 in that there are asymptotes at x = ±.8, whereas there are four asymptotes at the values where y = z is equal to .8829, 0489, –.1448, and –.7862. For a fixed value x>.8 therefore, Figure 9 shows solutions for y on branches 1 2 3 4. For –.8 < x < .8 there are solutions on branches 2 3 4 5. For x < –.8 there are solutions on branches 3 4 5 6.

13.3 Intersectons at x = ± .8, y = ± .2, and 2 = ± .4.

More in general, when we intersect F6 = 0 with a plane at fixed value of x > .8, or y > .2, or z > .4, branches 1 2 3 4 will appear. For x < –.8, or y < –.2, or z < –.4 branches 3 4 5 6 will be shown. And for the intermediate values of x or y or z, we will see branches 2 3 4 5. If we take x =.8, or y =.2, or z =.4, only three branches will appear: 2 3 4. And if we take x = –.8, or y = –.2 , or z = –.4, we shall see the branches 4 5 6.
The intersections of F6 = 0 with the six planes x = ±.8, y = ±.2, and z = ±.4 are shown in Figure 9. We come back to this figure in section 13.5.

13.4. F6 as function F6 = f(x,y,z)

We may look upon F6 as a funcion f(x,y,z). I.e., one might select any values of (x,y,z) and calculate F6 from this function. The value of F6 then will in general not be equal to 0. In fact, F6 as a function defines a surface in four dimensions, and the solutions for F6 = 0 just shows the intersection of this four-dimensional surface with the plane F6 = 0. In the same way as in section 7.6, the function can be re-written as

$$F6 = A_x x^2 + B_x x + C_x \qquad (1)$$

where A_x, B_x and C_x are functions of y and z.

Similarly:

$$F6 = A_y y^2 + B_y y + C_y \qquad (2)$$

$$F6 = A_z z^2 + B_z z + C_z \qquad (3)$$

Suppose we take a fixed value of z. At this intersection of F6 with the plane at the fixed value of z, we shall find the usual four branches. The coordinates of their aymptotes do not depend upon the value of F6.
E.g., for fixed value of z, two asymptotes can be found by solving for y from $A_x = 0$, and two other asymptotes by solving x from $A_y = 0$. Similarly for any fixed value of x, or of y. However , when F6 ≠ 0, we still may find asymptotes in this way, but the shape of the

branches might become much more different than with the case when F6 = 0. We come back to this aspect later.

13.5 Triangles and crosses

In the "perfect" example of section 12, where F6 can be factorized, it was shown that each factor contains a triangle, and in addition a "cross" (three orthogonal lines) through the middle of the triangle. The question is: can we find back such triangles and crosses in the imperfect example?

Firstly, can we find back "triangles"? The answer is: we can find them back approximately. One such triangle is formed by the intersections of branch 3 with the planes x = .8, y = .2, and z = .4, as shown in Figures 9A–C. The difference with the perfect case is that the sides of the triangle are curved, although they have straight lines as asymptotes. A second such triangle is shown in Figures 9D–F, where the fourth branch intersects planes at x = –.8, y = –.2, and z = –.4.

In analogy with the perfect case one should expect , on the basis of the first triangle, that branches 2 and 4 approximate a "cross" somewhere. This cross is identified in the first solution of Table 16, illustrated in Figure 10. The figure shows that branches 2 and 4 are very close to their asymptotes. But instead of intersecting each other (as they should do in a real cross), there is a small gap at the center, and branch 3 passes through this gap. The numerical rationale of the solution is that in the graph for given x, there must be asymptotes at the given values of y and z, and similarly for the graph of given y, or z. Such a solution is found at a point where (x y z) satisfies $A_x = 0$, $A_y = 0$, and $A_z = 0$ (remember that these equations identify the location of asymptotes).

The second solution of Table 16 is illustrated in Figure 11. This solution is less convincing. Branches 3 and 5 approximate a cross, and branch 4 passes through the gap in the center. But the gap is quite large. Nevertheless, the two solutions show that the characteristic feature of the perfect case can be found back, approximately, for the imperfect example.

We add a few remarks.

(i) There are many solutions for (x y z) such that $A_x = A_y = A_z = 0$. But they do not always produce approximate "crosses". E.g., there are solutions where in the upper-right part of the graphs the asymptotes apply to branches 3 and 5, whereas in the lower-left part they apply to branches 4 and 6.

(ii) In the perfect case, the center of the cross represents a triple singular point. I.e., if the origin is translated to that point , there should be three zero eigenvalues. For the given example, such a solution is not feasible. At best we could approximate a solution where second, third, and fourth eigenvalue of the reduced matrix M (i.e., after subtraction of (x y z) from its diagonal) are close to zero, which would indicate that branches 2 3 4 almost intersect. Such an approximation is found in Figure 10.

Similarly, a cross formed by branches 3 4 5 implies that there is a solution for (x y z) such that the reduced matrix has three positive eigenvalues, followed by three zero eigenvalues, and one negative eigenvalue. The solution in Figure 11 approximates such a requirement.

(iii) One might also try to find a solution for (x y z) where the three smallest eigenvalues of the reduced matrix are close to zero. Such a solution is suggested in the third example of Table 16: the asymptotes apply to branches 4 and 6, whereas branch 5 passes through the gap between them.

(iv) The perfect example in section 11 shows that the concept of "branch" is questionable. Figure 7 shows three perfect hyperbola's, and it is quite artificial to define 6 branches in terms of intersections with lines parallel to x = y. The idea of identifying "crosses" is not always compatible with the idea of identifying "branches", especially if branches intersect each other.

13.6 Saddle points of F6

The last remark in the previous section shows that we should consider the possibility of intersections between branches. We then have to go back to equations (1), (2), and (3) given in section 13.4.

Suppose we take fixed values of both y and z. Equation (1) then becomes the equation of a parabola: the equation of F6 as a quadratic function of x. At some value of x, F6 will have an extremum: the apex of the parabola. At this point it must be true that $F6 = -B_x x/C_x$ and the corresponding value of x must be exactly mid-way between the two values of x where F6 = 0 for the given y and z. Between branches 1 and 2, the apex has negative value for F6, between branches 2 and 3 positive value, between branches 3 and 4 negative, and so on. To illustrate further, select a fixed z > .4. The graph for this fixed z, shows branches 1 2 3 4. Between branches 2 and 3, the apex gives a maximum for any given y. But for some value of y, this maximum will have a minimum: we then have a minimax solution, which satisfies $G_x = G_y = 0$. This solution is located midway between branches 2 and 3, both in the horizontal and in the vertical direction. But the solution does not guarantee that $G_z = 0$. So there must be some value of z, where the minimax also satisfies $G_z = 0$. At this point the minimax obtains extreme value, and becomes either a miniminimax or a maximinimax, as the case may be. If the minimax obtains extreme value at a point where F6 = 0, its extremum must be a miniminimax (because between branches 2 and 3 it remains true that $F6 \geq 0$, so that a solution with F6 = 0 must represent a minimum). It follows that such a point is a singular point: branches 2 and 3 must intersect (the solution obeys F6 = 0, and also $G_x = G_y = G_z = 0$). But if the minimax obtains an extremum at a point where F6 > 0 there is only a "quasi singular point": branches 2 and 3 intersect there at a value where $F6 \neq 0$. The extremum of F6 = then may not be an absolute extremum, but a local extremum, a local miniminimax or a local maximinimax.

13.7 QSP solutions

Remember that an MB-solution was defined by the equality $G_x = G_y = G_z = 0$, under the proviso that these partial derivatives are not all equal to zero. If they are all equal to zero, they give a solution (x y z) that defines a quasi singular point QSP (we shall use this abbreviation regardless of whether F6 = 0 (real singular points) or not).

We have found four QSP solutions where F6 = 0. They indicate intersections between branches 2 and 3, between 3 and 4, between 4 and 5, and between 5 and 6. There is no such solution between branches 1 and 2. In addition we have found 7 other QSP solutions where

F6 ≠ 0: one between branches 1 and 2, and two between branches 2 3, or 3 4, and 4 5. Altogether, we found 11 QSP solutions. We believe that it can be proved (on the basis or arguments mentioned in section 7.7.2) that there are no other QSP solutions – but we cannot substantiate this belief by giving an actual proof. Table 17 gives a list of these 11 QSP solutions. The table also specifies their nature (as maximaximin, miniminimax, etc.), and whether they are absolute or local (solutions QSP2, QSP5, and QSP8, where F6 = 0, are absoute – solutions QSP1 and QSP11 have no competitive – the other six solutions are local).

13.8 Interpretation of QSP solutions

(i) Between branches 2 and 3, the apex of the parabola (defined by equation (1) of section 13.4 – and taking fixed valus of y and z) represents a maximum of F6 as function of x. For some value of y (keeping z fixed) we then obtain a minimax. And for some value of z this minimax becomes an absolute miniminimax (QSP2), whereas for other values of z we obtain a maximinimax (QSP3) or a minimaximax (QSP4). Similarly between branches 3 and 4. The apex of the parabola (for fixed y and z) is a minimum. For some value of y it becomes a maximin, when z is kept fixed. And for some value of z this maximin has an absolute maximum: maximaximin (QSP5), and at other values of z we find a local minimaximin (QSP6) or a maximaximin (QSP7).

(ii) But between branches 4 and 5 the situation is different. We there find QSP8 as an absolute miniminimax. And so we may expect a local miniminimax somewhere (QSP10), and a local maximinimax. However, QSP9 shows a local minimaximax – against expectations. The explanation is that between z = –.38 and z = –.72, there are three solutions for $G_x = G_y = 0$ (given z). This was illustrated before in Figure 3: this figure in fact shows the graph for x and y, given z = –.5. In the figure, there are three solutions where $G_x = G_y = 0$. Two of them are located where branches 4 and 5 are close together (minimax solutions, for given z = –.5), and the third one is located between these first two, where branches 4 and 5 are relatively far apart. This third solution represents a local maximax. And so, between branches 4 and 5, we find two solutions where the minimax has a minimum (QSP9 and QSP10), and one solution where the maximax has a minimum (QSP9).

(iii) This is further illustrated in Figure 12A, where z = –.6, and F6 = 0. This graph shows branches 3 4 5 6. The value of F6 is positive above branch 3, negative between 3 and 4, positive between 4 and 5, negative between 5 and 6, and positive below branch 6.

To dramatize the example, think of the figure as representing a mountainous landscape, with F6 = 0 as the water level. This means that between branches 3 and 4, and between 5 and 6, we have two lakes, separated by the mountainous landscape between branches 4 and 5: F6 gives the height of the mountains between branches 4 and 5. Suppose now that the water level is lowered to F6 = –.02. This means that the ford between branches 3 and 4 falls dry, and we obtain Figure 12B: the dry land above branch 3 becomes joined with the dry land between 4 and 5. The branches change of character, and we obtain three lakes instead of two. (Before we continue, two remarks. The first one is that for F6 = 0, the branches have at each point a tangential with negative slope. This no longer is true for F6 ≠ 0. The second remark is that change of the value of F6 does not affect the value of A_z: asymptotes remain the same.)

Suppose now that the water level is even more lowered: to F6 = –.3. We then obtain Figure 12C: the narrowest ford in the lake between branches 5 and 6 falls dry. The area of dry land in the middle becomes much larger: four lakes remain. On the other hand, suppose the water level is raised to F = .03. The land where the ridge between the two lakes is narrowest, becomes overflown, and the two lakes join. Branches 3 and 6 remain relatively the same (they shift a little away from the origin), but branches 4 and 5 change drastically: Figure 12D. Rising the water level even higher, to F6 = .04, has the effect that the peninsula in Figure 12D becomes an island: Figure 12E. And of course, at still higher water, this island, too, would be drowned.
The figures also show why between branches 4 and 5 there will be three solutions for given z = –.6, and $G_x = G_y$. The parabola's between these two branches have an apex with positive value: a maximum. In Figure 12A one may represent them by tops of mountains. In Figure 12D the mountain with lowest top is drowned: this mountain was a minimax. In Figure 12E the mountain with second highest top (minimax) is also drowned, but the maximax solution remains dry, as the middle of the island. But if we raise the water level even more, this mountain also will be overflooded.

13.9 MB-solutions

Six MB-solutions have been identified. The possibility that there are more of them cannot be entirely excluded. The solutions are listed in Table 18, together with the corresponding solutions for t, the values of the partial derivatives, and the eigenvalues of M after subtracting of (x y z) from its diagonal. The following comments can be given:

(i) The values of (x+y+z) are stationary values. They are found under the restriction $t_i't_i$, whereas eigenvalue solutions are stationary under the less demanding restriction t't = k. It follows that the values of (x+y+z) are in an interval bounded by the triples of the largest and of the lowest eigenvalue of M. These boundaries are 5.742921 and – 2.950905. MB1 is quite close to the upper boundary, whereas MB5 and MB6 approach the lower boundary.

(ii) After subtraction of the values (x y z) from the diagonal of M, six eigenvalues of the reduced matrix will be found, one of them equal to zero. The position of this zero eigenvalue determines on which branch the MB-solution is located.

(iii) One might have expected to find (at least) one MB-solution on each branch. However, no solution on branch 6 was found, whereas there are two solutions on branch 5. The explanation is rather simple. Figure 13 gives a graph of QSP11, where branches 5 and 6 intersect. By increasing the value of x, branch 5 becomes W-shaped. The upper left half of this W can be said to stem from one of the smooth curves at the point of intersection, the lower right half from the other. In fact, in Figure 13, if one plots the MB5 and MB6 points, one will see that they are (approximately) located on the W of branch 5, at opposite sides of the intersection. The "artificiality" of the concept of "branch" has been mentioned earlier.

(iv) In a perfect solution, where F6 = 0 can be factorized into 2 factors (when $m_i = 2$), we should find solutions for t such that all t_i are equal to either (±1 0) or to (0 ±1). Note that for MB1 such a "perfect" solution is almost realized. And for the other MB-solutions, too, we find many examples where t_i approaches (±1 0) or (0 ±1); but there is no systematic pattern.

(v) Figures 14 to 19 give graphs of the six MB-solutions. They show that these solutions are of different kind. This can be specified by taking projections on the line x = y = z (the Appendix discusses the meaning of such projections). In the graphs for MB1 the projections show a minimum. The same applies to MB2. With MB3 the situation is different. In the (x z) and (y z) planes, MB3 is a local maximum. In the (x y) plane MB3 appears to be located on a bending point of branch 3: the projection is neither maximum nor minimum. MB4 is a local minimum in the (x y) and in the (y z) plane. But in the (x z) plane its projection has a maximum. It therefore is a saddle-point, in terms of its projections. This is also true for MB5 and MB6 with local minima in the planes of (x y) and (x z), but a maximum in plane (y z).

(vi) What is the relation between MB and QSP solutions? An MB-solution requires that F6 = 0, whereas the partial derivatives must be equal, but not all equal to zero.
A QSP solution requires that all partial derivatives are equal to zero, but do not require that F6 = 0. It follows that an MB-solution must be close to a QSP solution, if for the MB-solution the partial derivatives are close to zero. In the present example, partial derivatives at MB1 are not close to zero. It follows that we should not expect a QSP solution close to MB1. For the other five MB-solutions the partial derivatives are close to zero (although it remains debatable what "close" means). And so we might expect these other five MB-solutions to be close to a QSP solution. In fact MB2 is close to QSP4. That MB2 must be close to some QSP point also can be derived from the eigenvalues of the reduced matrix (M after subtraction of (x y z) from its diagonal). The second eigenvalue must be zero, by definition. But we find that the third eigenvalue is close to zero, which indicates that at MB2 we are not far away from branch 3. In other words, we are not far away from a point where branches 2 and 3 intersect (solution QSP4). Similarly, in solution MB3 the third eigenvalue of the reduced matrix equals zero, but second and third eigenvalue are also close to zero. MB3 therefore should not be too far away from a QSP point between branches 2 and 3 (it is QSP4 again), and from a QSP point between branches 3 and 4 (QSP7). MB4 shows up with third and fifth eigenvalue not so very close to zero. If anything, MB4 is close to QSP10 (between branches 4 and 5). MB5 and MB6 have zero eigenvalue at the fifth position. But the sixth eigenvalue is close to zero, so that we may expect MB5 and MB6 to be close to a QSP point between branches 5 and 6: QSP11.

(vii) Conversely, if we have a QSP solution with F6 = 0, should we expect to find an MB-solution in its neighbourhood? The answer is negative. In its neighbourhood one may find other solutions where all partial derivatives are equal, but then with F6 different from zero: this does not give an MB-solution (which requires F6 = 0). A QSP point, of course, has its own tangential planes (or hyperplanes). Their specifications depend on higher order derivatives of F6. It might happen that at a real singular point (QSP with F6 = 0) such a tangential plane is orthogonal to the line x = y = z. Such a QSP would be an MB-solution itself. It would be a rare happening, and in the example QSP2, QSP5, or QSP8 do not even approximate such an exceptional finding.

14 CONCLUSIONS

This monograph confirms that eigenvalue solutions can be rather "unfair". They satisfy the requirement $t't = k$, but the subvectors t_i may have quite different norm $t_i't_i$. This implies that an eigenvalue solution may be dominated by only some of the k sets, while ignoring the other sets. A possible remedy could be to take solutions which require $t_i't_i = 1$, called MB-solutions. When I wrote the 1984 paper, it was not entirely clear to me why in the example used there (the same as in section 7), bending points appear in the third branch – with the consequence that there is more than one MB-solution on that branch. Of course, I was aware of the ambiguity in the concept of "branch": in a perfect case (as in section 11) one obtains a graph with perfect hyperbola's which may intersect each other in many places. The concept of "branch" then becomes quite artificial. At such intersection points, the partial derivatives of the characteristic function will be all equal to zero. At MB-solutions, partial derivatives should be equal to each other, but not equal to zero. It follows that there is an intimate connection between QSP solutions and MB-solutions, especially if for the latter it would be true that the partial derivatives are equal to each other, and also very close to zero. The present monograph explains why bending points may appear in branches, as a result of the presence of QSP points. It also shows that MB-solutions may be very close to QSP solutions, especially to QSP solutions where the characteristic function F is not exactly equal to zero. The monograph shows that the surface defined by the characteristic function can be a tricky one: there is a tendency that the MB or QSP solutions represent "saddle points". This implies that algorithms to solve for MB or QSP solutions may meet various special difficulties (may converge in a wrong way, may not converge but remain oscillating between branches, may require extremely good initial estimates). The monograph does not offer effective algorithms, but may give some sugestions for the development of such algorithms. Finally, the question remains: to what extent are MB-solutions useful in practice? I have no general verdict. MB-solutions have their own peculiarities. But any method of data analysis has its own peculiarities. The relevant question is : are the peculiar properties of the method tuned to the peculiar properties of the emprical data?

APPENDIX. REPRESENTATION OF t'Bt

A1 t'Bt and t'Mt

Firstly, for any t (assume t't = k) the value t'Bt must be non-negative, because t'Bt is the variance of the vector PΦt. On the other hand, t'Mt may be negative, because M = B–I so that t'Mt = t'Bt–k.

A2 Relations between F = 0 and the line x = y

For simplicity, the further discussion will be restricted to the case where k = 2, and $m_1 = m_2 = 2$, as in the example of section 7. Figure 1 shows that the intersection of F = 0 and the line x = y gives the eigenvalues. Figure 2 implies that the line x = y is an asymptote of $G_d = G_y - G_x = 0$. Take a point $(\underline{x},\underline{y})$ on any branch of F = 0. Its tangential intersects the line x = y at a point with coordinates equal to t'Mt/k.

Proof. The tangential is the line

$$(x-\underline{x})G_x + (y-\underline{y})G_y = 0 \text{ or, equivalently}$$

$$t_1't_1x+t_2't_2y = t_1't_1\underline{x} + t_2't_2\underline{y}.$$

The term at the right is equal to t'Mt for the given values $(\underline{x},\underline{y})$. It follows that at the intersections with x = y, we have $x = y = t'Mt/(t_1't_1 + t_2't_2) = t'Mt/k$.

It also follows that the intersection has (signed) distance to the origin (x y) = (0 0) equal to $(t_1'Mt/k).k^{1/2} = t'Mt/k^{-1/2}$.

Moreover, the distance to the origin of the original axes μ_1 and μ_2 – this is the point (x y) = (–1 –1) – will be equal to $t'Bt.k^{-1/2}$. This distance is always positive.

(i) To illustrate, take point E1 in Figure 1, with coordinates equal to the eigenvalue 1.340 in Table 6. By definition, this eigenvalue will be equal to t'Mt/k. The distance to the origin (0 0) will be equal to $(1.340).2^{1/2} = 1.895$. The distance to the point (–1 –1) is equal to $(2.340).2^{1/2} = 3.309$.
Or take the point E4. Its distance to (0 0) is equal to $(-.898).2^{1/2} =-1.269$, a distance with negative sign. But the distance to (–1 –1) is positive:
$(1-.898).2^{1/2} = (.102)2^{1/2} = .145$.

(ii) As another illustration, take the point MB1. For this pont we have t'Mt = x+y = 2.554 (Table 7). The tangential intersects the line x = y at a point with coordinates 2.554/2 = 1.277 = t'Mt/k. The value of t'Bt/k then is equal to (1.277+1) is 2.277. Since the tangential at an MB point has slope –1, its intersection with the line x = y (this line has slope +1) is also the *projection* of the MB point on x = y. In Figure 1 it can be seen that this projection is a minimum: other points on the first branch have larger projections. In terms of the value of t'Mt, hoewever, MB1 is neither minimum nor maximum: points above MB1 have tangentials that intersect x = y at lower value, and points below MB1 have tangentials that intersect x = y at larger value (the value obtains a maximum at the point E1, and then decreases again, so that somewhere at the right of E1there will be a point where the tangential has the same intersection with x = y as the tangential of MB1.)

Clearly, MB1 has been defined as a *conditional* stationary value: the special condition is $t_1't_1 = t_2't_2 = 1$ (or, at an MB point the tangential must have slope –1). Given this special condition, MB1 gives a maximum for t'Mt (other MB points have smaller projections on x = y). In the same way, MB6 stands for a conditional minimum of t'Mt. But in terms of projections on the line x = y, MB6 is a maximum (other points on the fourth branch have smaller projections). And in terms of intersections between a tangential and the line x = y, MB6 is neither a minimum nor a maximum.

(iii) For eigenvalue solutions E1 to E4 it also is true that the intersection of their tangential with the line x = y is the same as the projection on this line: simply because such a point is located on the line itself. In Figure 1, the point E1 gives a maximum for t'Mt, because points in the neighbourhood of E1 have tangentials with smaller intersection with the line x = y. In terms of projections on this line, however, E1 is neither maximum nor minimum.

A3 Interpretation of eigenvalue solutions

(i) In section 7 we used the upper-left 4 x 4 minor of Table 1 for the numerical example. Let us call this 4 × 4 matrix: $\underline{B}$. Take solutions for t such that t't = k. Then t'$\underline{B}$t is proportional to the squared lenght of the vector $P\Phi t = P_1\Phi_1 t_1 + P_2\Phi_2 t_2$. Vectors $P\Phi t$ will be located on the surface of a four-dimensional hyperellipsoid. Eigenvalue solutions define the principal axes of this hyperellipsoid. The first eigenvalue gives an unconditional maximum for t'$\underline{B}$t, the last eigenvalue gives an unconditional minimum. Intermediate eigenvalues give neither a maximum nor a minimum in the absolute sense. E,g., the second eigenvalue gives a maximum for t'Bt for a solution of t restricted to the null-space of the first eigenvector, and gives a minimum in the null-space of the last two eigenvectors.

(ii) The equation F12 = 0 imposes additional restrictions upon solutions for t. They imply that vectors $P\Phi T$ cannot be everywhere on the surface of the hyperellipsoid, but are restricted to a certain "path" upon that surface, a path that includes the eigenvalue solutions.

This explains why in Figure 1 we find E2 as a maximum. I.e., in the immediate neighbourhood of E2, other vectors on the "path" defined by F12 = 0 have smaller value for t'$\underline{B}$t. But other vectors in the immediate neighbourhood *not* on the path may have larger values for t'$\underline{B}$t (e.g., if such a vector deviates from E2 in the direction of E1). Similarly for E3. In Figure 1, this eigenvalue solution also appears as a maximum: the corresponding value of t'$\underline{B}$t is larger than for vectors in the immediate neighbourhood of E3, as long as these vectors are on the path defined by F12 = 0. Outside this path, other vectors in the immediate neighbourhood of E3 may have larger values for t'$\underline{B}$t.

A4 Interpretation of bending point solutions

In section 7.7.3 the two bending points B1 and B2 were mentioned "for completeness" only. We now see that they are solutions in their own right. Table 19 shows that B1 and B2 obtain minimum values for t'$\underline{B}$t, as long as we follow the third branch of F12 = 0. This can be visualized as follows: t'$\underline{B}$t is proportional to the distance between points on the surface of the hyperellipsoid and the origin. If solutions for t agree with F12 = 0, values of t'$\underline{B}$t are proportional to such distances only for points on the surface lcated on the path described by F12 = 0, whereas points on the surface outside this path are ignored. Under this restriction, E2 becomes a maximum, and so does E3. At the same time, B1 and B2 give minima for t'$\underline{B}$t, but only compared to points in the immediate neighbourhood located on the path (not compared to points outside the path).

In section 4 it was demonstrated that F12 = 0 applies to solutions with any apriori fixed value of $c_1 = t_1't_1$ (as long as $c_1 \leq k = 2$). If one selects $c_1 < .801739$ (which means that the tangential has slope > −1.494677) there will be *four* solutions for stationary values. Similarly if one selects $c_1 > 1.382273$ (which means tangential slope <.446422). But between these two given boundaries (they correspond with B1 and B2) there are *six* solutions (such as for $t_1't_1 = t_2't_2 = 1$). And at B1 and B2 themselves we have minima for t'$\underline{B}$t, as long as solutions are restricted to the path on the surface of the hyperellipsoid defined by F12 = 0. On the other hand, solutions B1 and B2 tend to be "unfair". For solutions on the third branch in Figure 1, the point B1 has steepest tangential (is dominated by the first set more than at neighbouring points), whereas B2 has flattest tangential (is dominated by the first set, more than at solutions in the immediate neighbourhood of B2).

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TABLES

Table 1 Matrix B = ΦP'PΦ/n to be used as illustration

1.8	0	.7	.4	.7	.1
0	.2	.1	.2	.3	.2
.7	.1	1.2	0	.5	.4
.4	.2	0	.8	.1	.3
.7	.3	.5	.1	1.4	.0
.1	.2	.4	.3	0	.6

Table 2 Matrix M = B – I

.8	0	.7	.4	.7	.1
0	–.8	.1	.2	.3	.2
.7	.1	.2	0	.4	.5
.4	.2	0	–.2	.1	.3
.7	.3	.5	.1	.4	0
.1	.2	.4	.3	0	–.4

Table 3 Matrix DB

μ_1	0	0	0	0	0
0	μ_1	0	0	0	0
0	0	μ_2	0	0	0
0	0	0	μ_2	0	0
0	0	0	0	μ_3	0
0	0	0	0	0	μ_3

TABLE 4 Matrix DM

x	0	0	0	0	0
0	x	0	0	0	0
0	0	y	0	0	0
0	0	0	y	0	0
0	0	0	0	z	0
0	0	0	0	0	z

TABLE 5 Eigenvalues μ_i of B (Table 1) and $(\mu_i - 1)$ of M (Table 2)

μ_i	2.914307	1.063080	.969963	.877752	.158522	.016375
$\mu_i - 1$	1.941307	.063080	–.030037	–.122248	–.841478	–.983625

TABLE 6 Eigenvector solutions for sets 1 and 2 of Table 2. For these solutions : x = y, F12 = 0, and $t_1't_1 + t_2't_2 = 2$.

x = y	1.339879	–.046182	–.396083	–.897613
G_x	3.705799	–.024174	.124668	–.806292
G_y	1.677944	–.388754	.179968	–.149156
$t_1't_1$	1.376663	.117086	.818473	1.687777
$t_2't_2$	.623337	1.882913	1.181527	.312223
t_1	1.171617	.313040	–.692365	.223499
	.063093	.138127	.582335	1.279774
t_2	.725020	–.946275	.715391	–.259132
	.312539	.993725	.818371	–.495052

TABLE 7 MB-solutions for first two sets of Table 2. For these solutions it is valid that $t_i't_i = 1$, that F12 = 0, that $G_x = G_y = G_i$

	MB1	MB4	MB2	MB5	MB3	MB6
branch	1	2	3	3	3	4
x	1.598612	–.668818	–.014085	–.695256	–.536487	–1.040253
y	.955084	.015878	–.686650	–.032358	–.283343	–.572661
x+y	2.553696	–.6529396	–.700735	–.727614	–.819830	–1.612914
G_1	2.060001	–.017947	.408500	.016815	.095117	–.258199
t_1	.997639	–.021200	–.963789	–.458716	–.568897	.365777
	.068738	.999966	.266689	.888341	.822412	.930705
t_2	.933950	–.461812	.730834	.999617	.653780	–.451832
	.357393	.886757	.682546	–.034581	.756681	–.892100

TABLE 8 Saddle points of F12 = f(x,y)

	maximin	minimax	maximin
x	1.723739	–.681748	–.918912
y	.361737	–.007922	–.399297
F12	–.549331	.000325	–.033233

TABLE 9 MB-solutions in "diagonalized" version

	MB1	MB2	MB4	MB6
x	1.5688	.0312	–.4699	–1.1301
y	.9688	–.5688	.1301	–.5302
	1	1	0	0
	0	0	1	–1
t	1	–1	0	0
	0	0	1	–1

TABLE 10 Relations between first and third set of Table 2

			EIGENVALUES			
x = z	1.3488		–.0535		–.3323	–.9630
			MB-solutions			
branch	1	2	2	2	3	4
x	1.5125	.1415	.1061	–.3919	.2248	–1.1136
z	1.1369	–.2345	–.1988	.2240	–.4347	–.6378
x+z	2.6494	–.0938	–.0937	–.1679	–.2100	–1.7514

TABLE 11 Relations between second and third set of Table 2.

		EIGENVALUES		
y = z	.8823	.0488	-.1449	-.7862
		MB-solutions		
y	.8167	–.0459	–.2134	–.6176
z	.9412	.1109	.0326	–.9131
y + z	1.7580	.0650	–.1808	–1.5307

TABLE 12 k = 2, m1 = m2 = 3, M12 is a permutated diagona; matrix

.8				.3	
	−.6		.1		
		−.2			.2
	.1		.3		
.3				−.2	
		.2			−.1

TABLE 13 MB-solutions for data given in Table 12

x	1.1	.5	−.5	−.7	0	−.4
y	.1	−.5	.4	.2	.1	−.3
	1	1	0	0	0	0
t_1	0	0	1	1	0	0
	0	0	0	0	1	1
	0	0	1	−1	0	0
t_2	1	−1	0	0	0	0
	0	0	0	0	1	−1

TABLE 14 MB-solutions for the perfect example with k = 3

	1	1	1	−1
w	1	1	−1	1
	1	−1	1	1
x	p+q	p−q	−p+q	−p−q
y	p+r	p−r	−p−r	−p+r
z	q+r	−q−r	q−r	−q+r
x+y+z	2(p+q+r)	2(p−q−r)	2(−p+q−r)	2(−p−q+r)
δ|H|	−pq−pr−qr	pq+pr−qr	pq−pr+qr	−pq+pr+qr

TABLE 15 Eigenvalues (μ–1) of matrix M (Table 2), with corresponding eigenvectors t (normalized to t't = k = 3) and values of $t_i't_i$ (which add up to k = 3) and values of G_x, G_y and G_z (proportional to those of $t_i't_i$).

μ–1	1.914307	.063080	–.030037	–.122248	–.841478	–.983635
	1.1870	.2810	–.9272	.6243	.4579	.2302
	.1671	.0547	.2782	–.6687	.2319	1.5461
	.8051	.1165	1.1272	.6417	–.8032	.1037
t	.3164	.9988	–.4212	–.9221	–.8990	–.2576
	.8685	–1.1020	.1019	–.8904	.0817	–.4706
	.2459	.8321	.7774	–.3289	1.1299	–.5079
$t_1't_1$	1.4369	.0819	.9373	.8369	.2635	2.4435
$t_2't_2$	.7483	1.0113	1.4479	1.2622	1.4533	.0771
$t_3't_3$	.8148	1.9068	.6147	.9010	1.2832	.4794
G_x	28.0392	–.000827	.004027	–.106026	.018155	–.288477
G_y	14.6021	–.010197	.006247	–.009051	.088467	–.009107
G_z	15.9000	–.019223	.002644	–.006482	.088467	–.056581

TABLE 16 Solutions for "crosses"

		graph of x		graph of y		graph of z	
		lowest	highest	lowest	highest	lowest	highest
asymp. of	x	–	–	-.694463	-.247870	-1.581522	-.247870
	y	-.496370	-.033378	–	–	-.330980	-.033378
	z	-.341112	.099406	-1.256056	.099406	–	–
asymp. of	x	–	–	-.645259	1.931421	-.645259	.181678
	y	-.128751	.002124	–	–	-.128751	1.727599
	z	-.274029	.768181	-.274029	.430420	–	–
asymp. of	x	-	-	-.982604	-.056282	-.982604	.428621
	y	-.646361	-.033378	-	-	-.646361	-.070417
	z	-.885837	.007910	-.885837	.177377	-	-

TABLE 17 QSP-solutions for F6 ($G_x = G_y = G_z = 0$)

		between branches 1 and 2	
		QSP 1	
x		2.510727	
y		.996486	
z		1.245795	
		maximaximin	
F6		-6.080870	
		between branches 2 and 3	
	QSP2	QSP3	QSP4
x	-.627702	-.495598	-.138815
y	-.001565	-.002070	.037629
z	4.326615	1.148768	.079407
	miniminimax abs	maximinimax loc	miniminimax loc
F6	0	1.631E-3	7.488E-5
		between branches 3 and 4	
	QSP5	QSP6	QSP7
x	.322785	-.026595	-.202066
y	-.073270	-.073385	-.088133
z	-.526628	-.153908	.022414
	maximaximin abs	minimaximin loc	maximaximin loc
F6	0	-4.364E-4	-3.452E-4
		between branches 4 and 5	
	QSP8	QSP9	QSP10
x	.223476	-.272685	-1.465632
y	-3.938299	-.742768	-.253472
z	-.433528	-.598644	-.019304
	miniminimax abs	minimaximax loc	miniminimax loc
F6	0	5.087E-2	9.724E-3
		between branches 5 and 6	
		QSP 11	
x		-1.005007	
y		-.747892	
z		-.931059	
		miniminimax	
F6		0	

TABLE 18 Six MB-solutions with $t_i't_i = 1$, F6 = 0, and $G_x = G_y = G_z$

BRANCH 1	BRANCH 2	BRANCH 3	BRANCH 4	BRANCH 5	BRANCH 5
Solutions for (x, y, z) and their sum					
2.313673	–.146294	–.109603	–1.154373	–.894927	–.904395
1.584949	.054842	–.009549	.268099	–.987831	–.592133
1.699601	.116530	.040201	–.252031	–.660107	–1.119583
5.598223	.025078	–.078951	–1.138305	–2.542865	–2.616066
Solutions for vectors t, with $t_i't_i = 1$					
.9866	–.9352	–.8921	–.4358	.4589	–.2616
.1630	.3541	.4518	–.9001	.8885	.9652
.9398	.6791	.9888	.9956	–.5911	.8153
.3417	–.7340	.1494	–.0934	–.8066	.5790
.9569	.9985	–.0578	.1334	–.2004	–.3764
.2905	.0556	.9983	.9910	.9797	–.9265
Solutions for $G_x = G_y = G_z$					
15.7977	–.003393	.002514	–.053327	.076587	.053928
Eigenvalues of M after subtraction of (x y z) from diagonal					
0	1.946	1.960	2.621	2.776	2.815
–1.597	0	.048	.681	.952	1.069
–1.702	–.042	0	.249	.824	.907
–2.112	–.154	–.103	0	.688	.549
–2.558	–.854	–.836	–.424	0	0
–3.227	–.946	–.914	–.850	–.153	–.108

TABLE 19 Points on the third branch of F12

	x	y	t'Bt/k	$t_1't_1$	$t_2't_2$
MB2	-.014085	-.686650	.649633	1	1
B1	-.325081	-.444128	.603594	.801739	1.198261
E3	-.396083	-.396083	.603917	.818473	1.181527
MB3	-.536487	-.283343	.590085	1	1
B2	-.663298	-.089567	.513778	1.382723	.617277
MB5	-.695256	-.032358	.636193	1	1

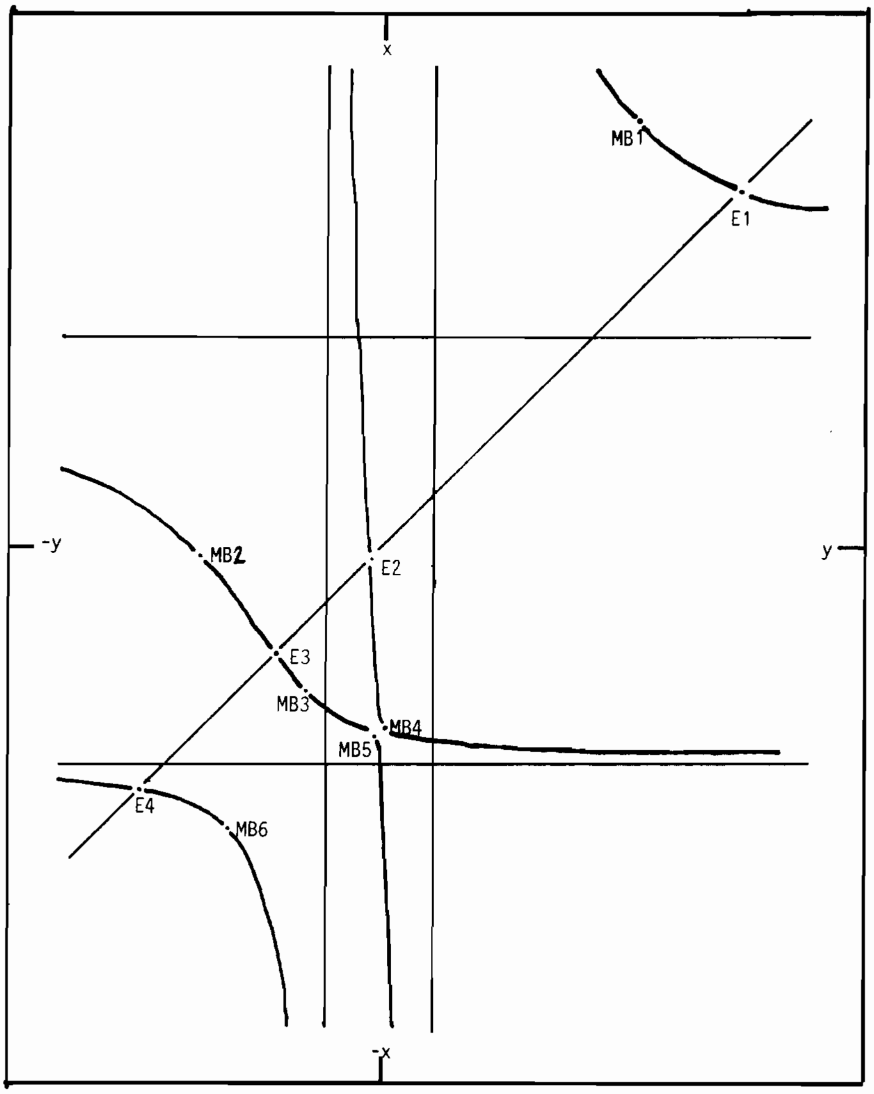

FIGURE 1 Graph of F12 = 0. Four eigenvalues (E1 to E4) are the intersections with line x = y. Six solutions MB1 to MB6 where tangential has slope -1.
Vertical axis: x, cut off at ± 1.8;
horizontal axis: y, from -1.2 to 1.6.

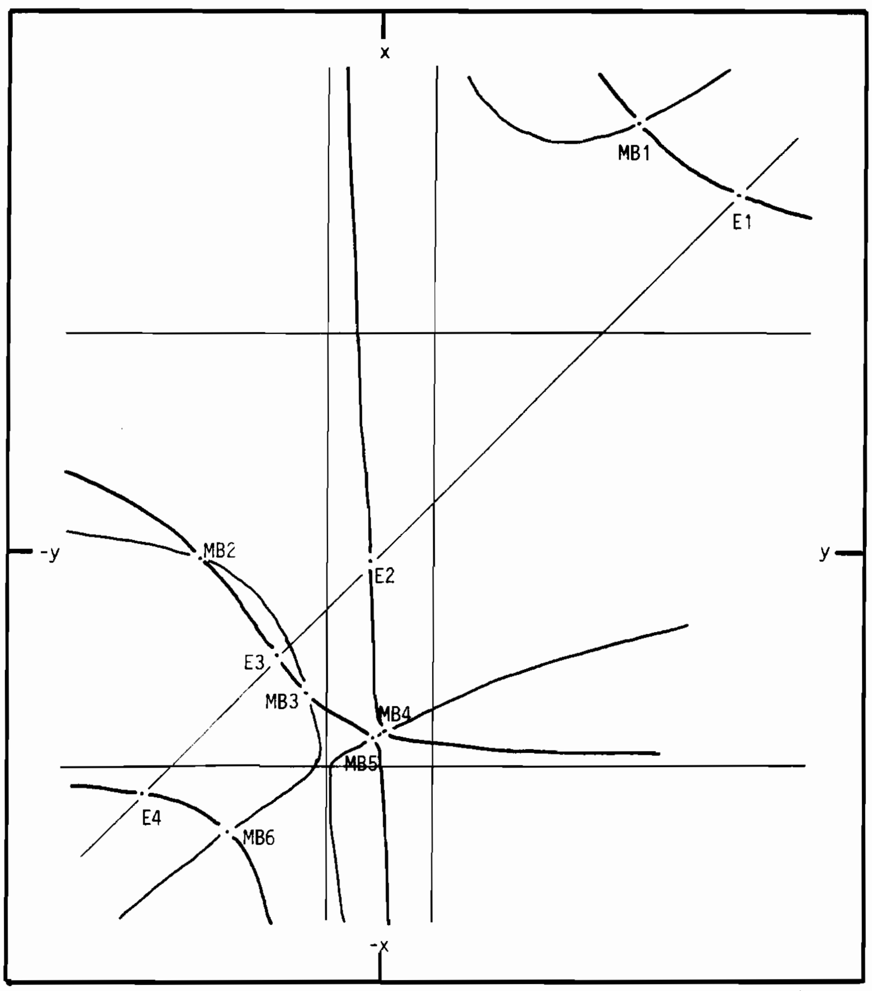

FIGURE 2 Graph of F12 = 0, as in Figure 1, but the six MB solutions now identified as intersections with the three branches of $G_d = G_y - G_x = 0$.
Vertical axis: x, from -1.4 to 1.8.

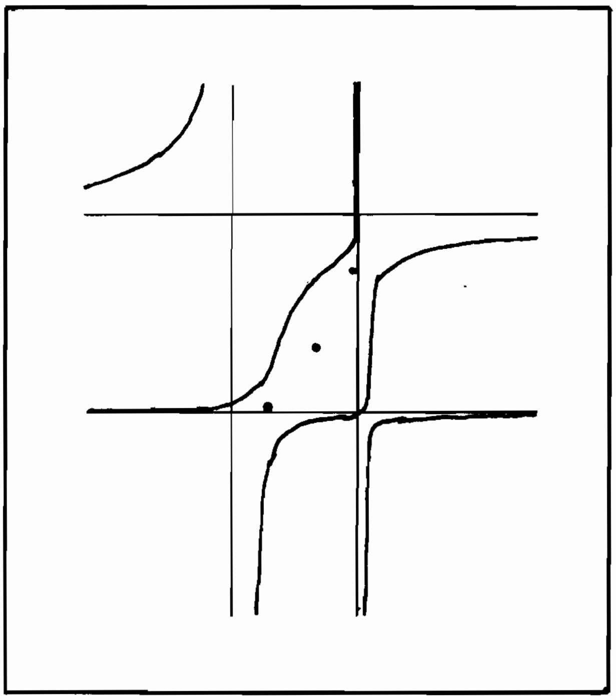

FIGURE 3 Example with three quasi singular points between middle two branches. There is also a QSP betweeen each of the two outer branches (but not indicated in the figure).

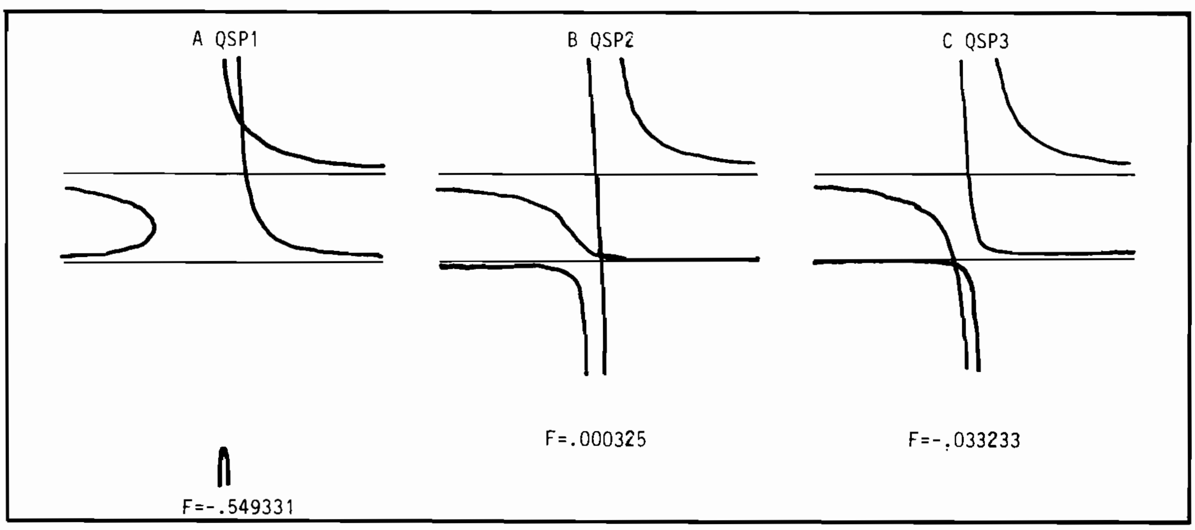

FIGURE 4 Graphs for the QSP of F12. Graph A for QSP1, with F12 = -.549; graph B for QSP2 with F12 = .000325, and graph C for QSP3 with F12 = -.033233.

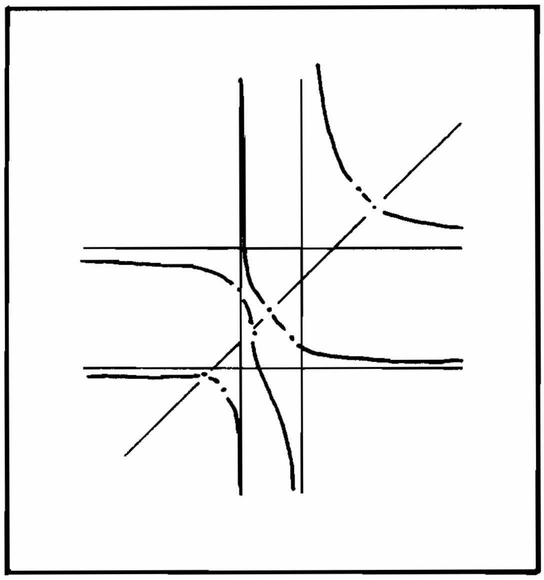

FIGURE 5 Graph of F13 = 0, with four eigenvalues on the line x = y, and four MB solutions, one on each branch.

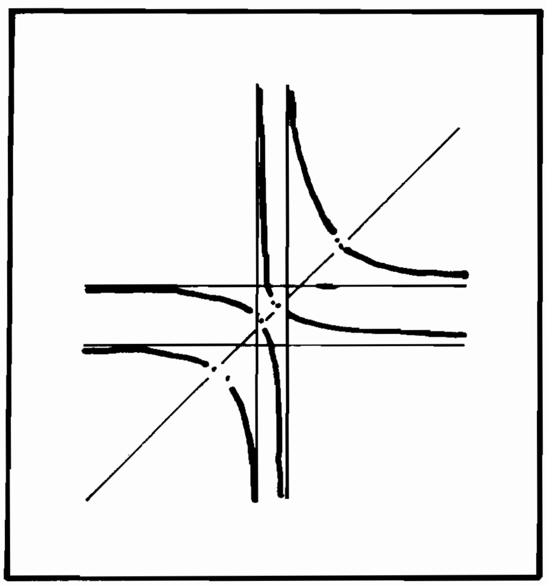

FIGURE 6 Graph of F23 = 0, with four eigenvalues on line x = y, and four MB solutions very close to the eigenvalues.

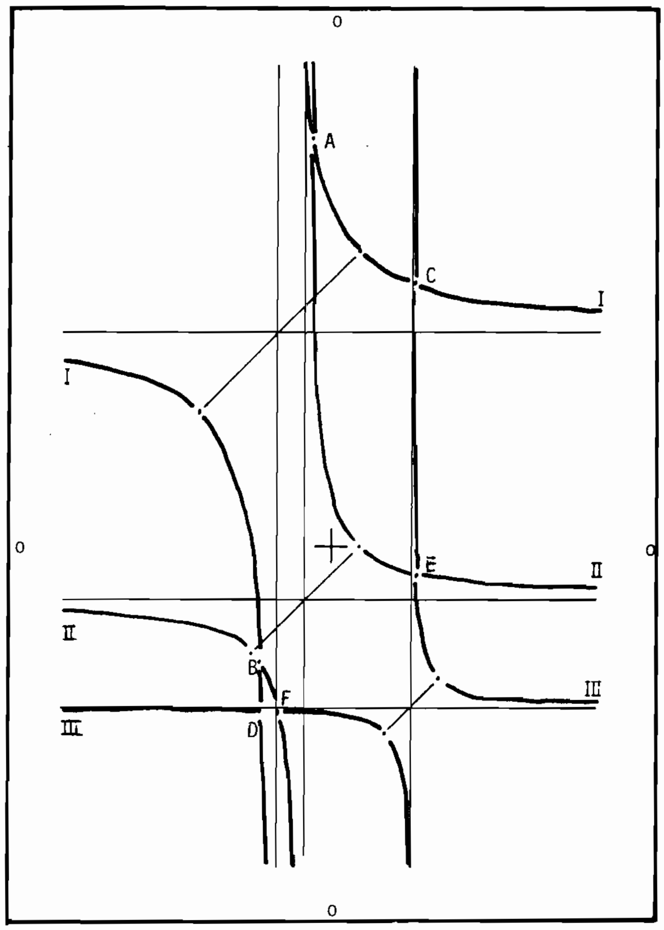

FIGURE 7 Graph of the characteristic function of Table 12. The graph falls apart into three hyperbola's:
I with symmetry about
x = .8, y = -.2,
II with symmetry about
x = -.2, y = -.1,
III with symmetry about
x = -.6, y = .3.
Hyperbola's I and II intersect in points A and B, I and III in points C and D, II and III in points E and F.

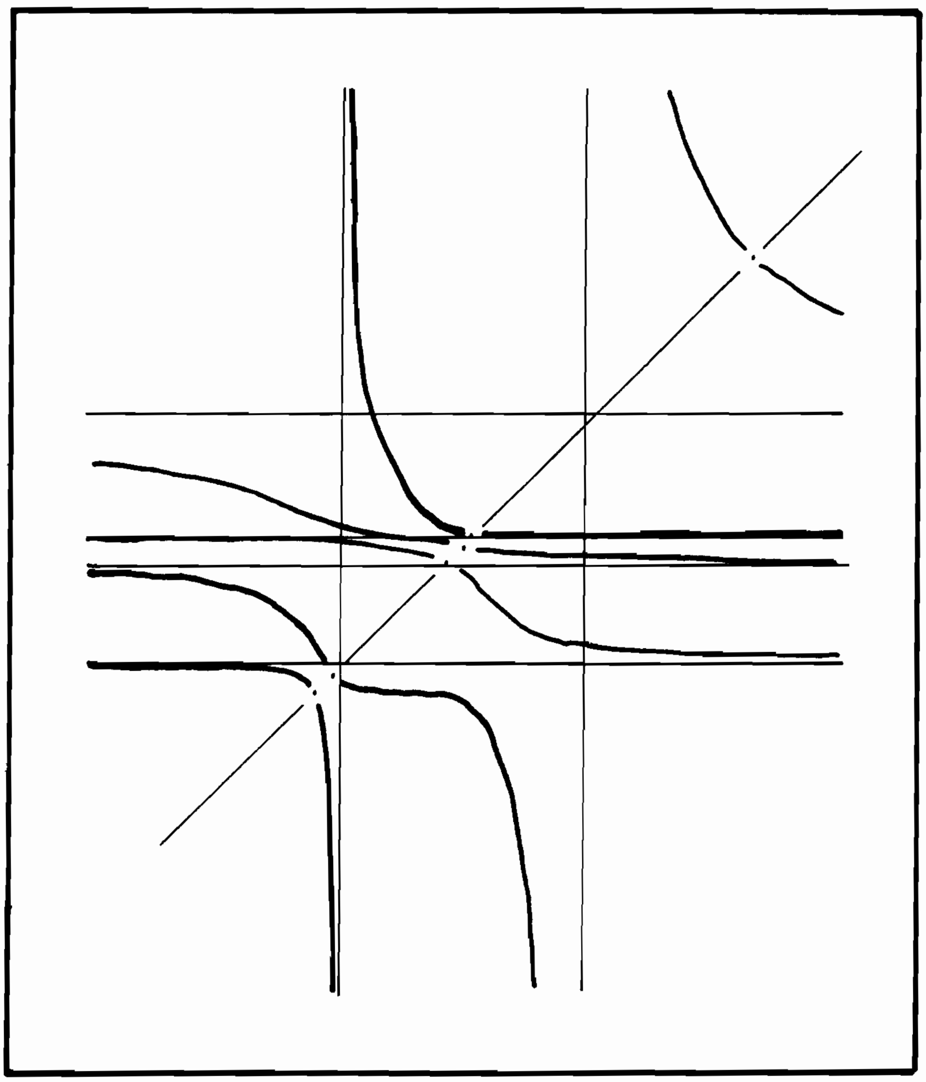

FIGURE 8 Graph of the intersection of F6 = 0 with the plane defined by x (horizontal axis) and y = z (vertical axis). Vertical asymptotes at x = ± .8. Horzintol asymptotes at y = .8829, .0489, -.1408, and -.7862. Six eigenvalues where the branches are intersected by y = x.

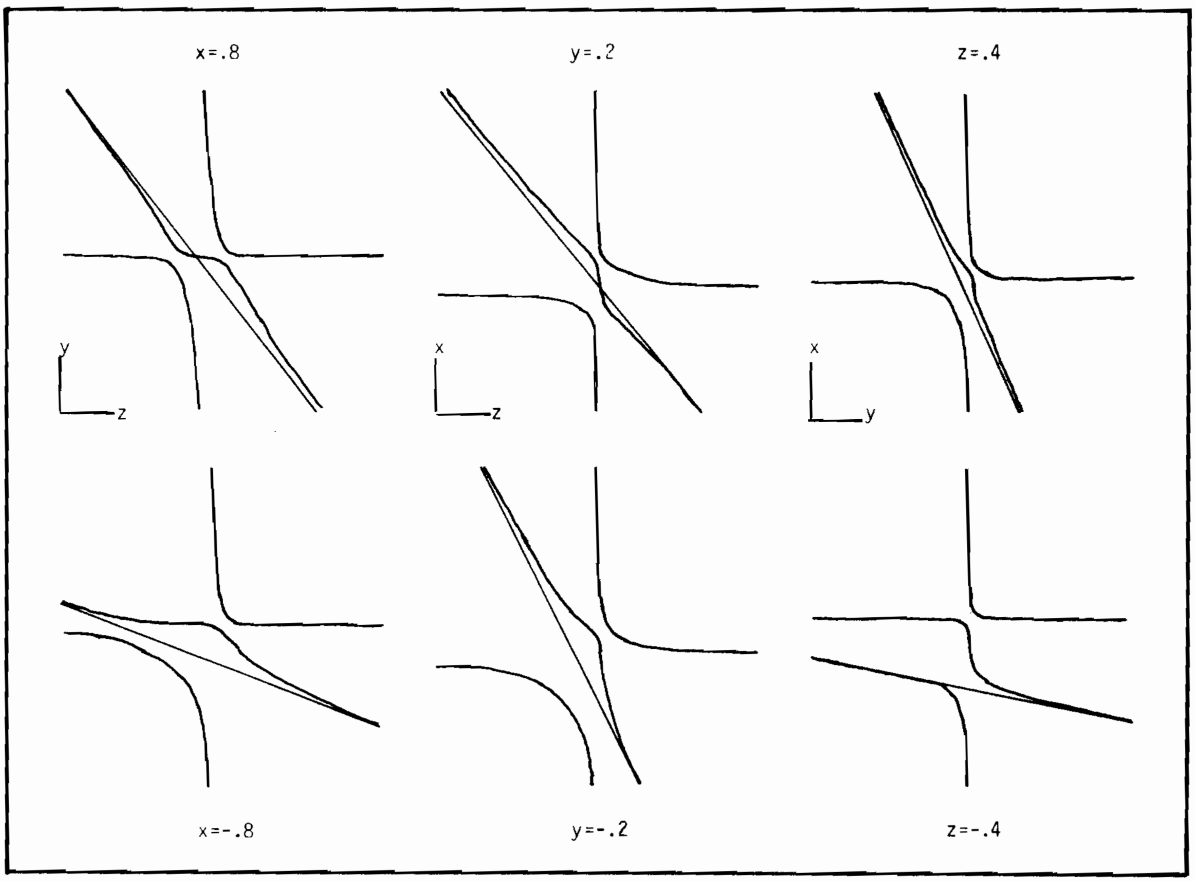

FIGURE 9 Intersections of F6 = 0 and the planes x = ± .8, y = ± .2, and z = ± .4.
In the upper three graphs branches 2 3 4 appear, in the lower three the branches 3 4 5. The middle branch has asymptote with negative slope. However, in the graph for z = -.4, it looks as if at the left the asymptote applies to branch 5. By taking increasingly more negative values of y, it will appear that branch 5 has a horizontal asymtote (the same as that of branch 3 for large y), whereas branch 4 obtains a more negative slope and approximates the oblique asymptote.

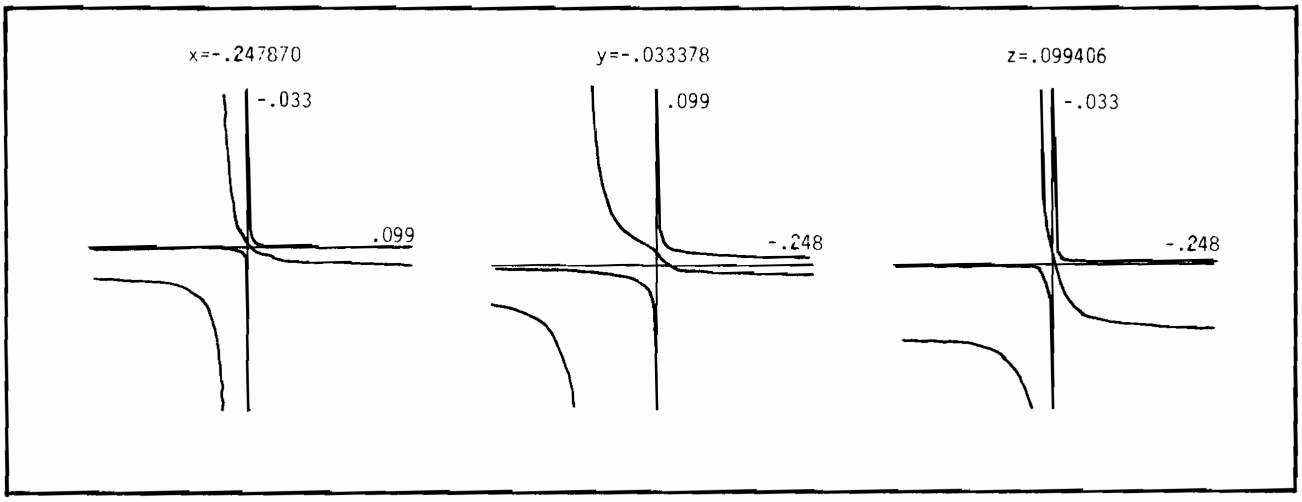

FIGURE 10 Graphs for the first "cross" in Table 16, with branch 3 passing through the "gap" between branches 2 and 4.

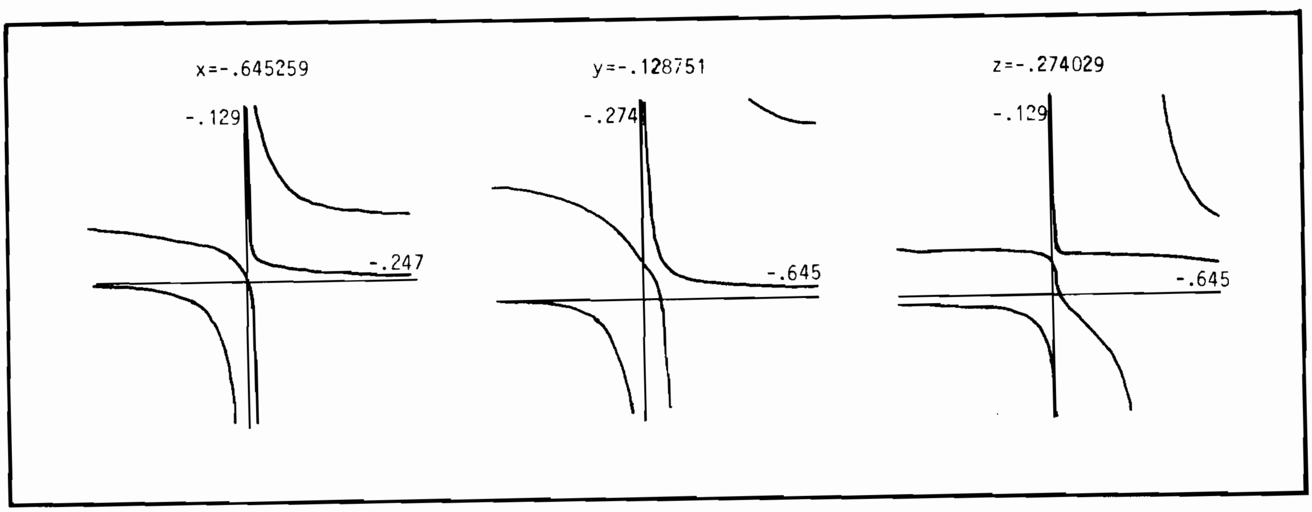

FIGURE 11 Graphs of second "cross" in Table 16, with branch 4 passing through the "gap" (much larger gap than in Figure 10) between branches 3 and 5.

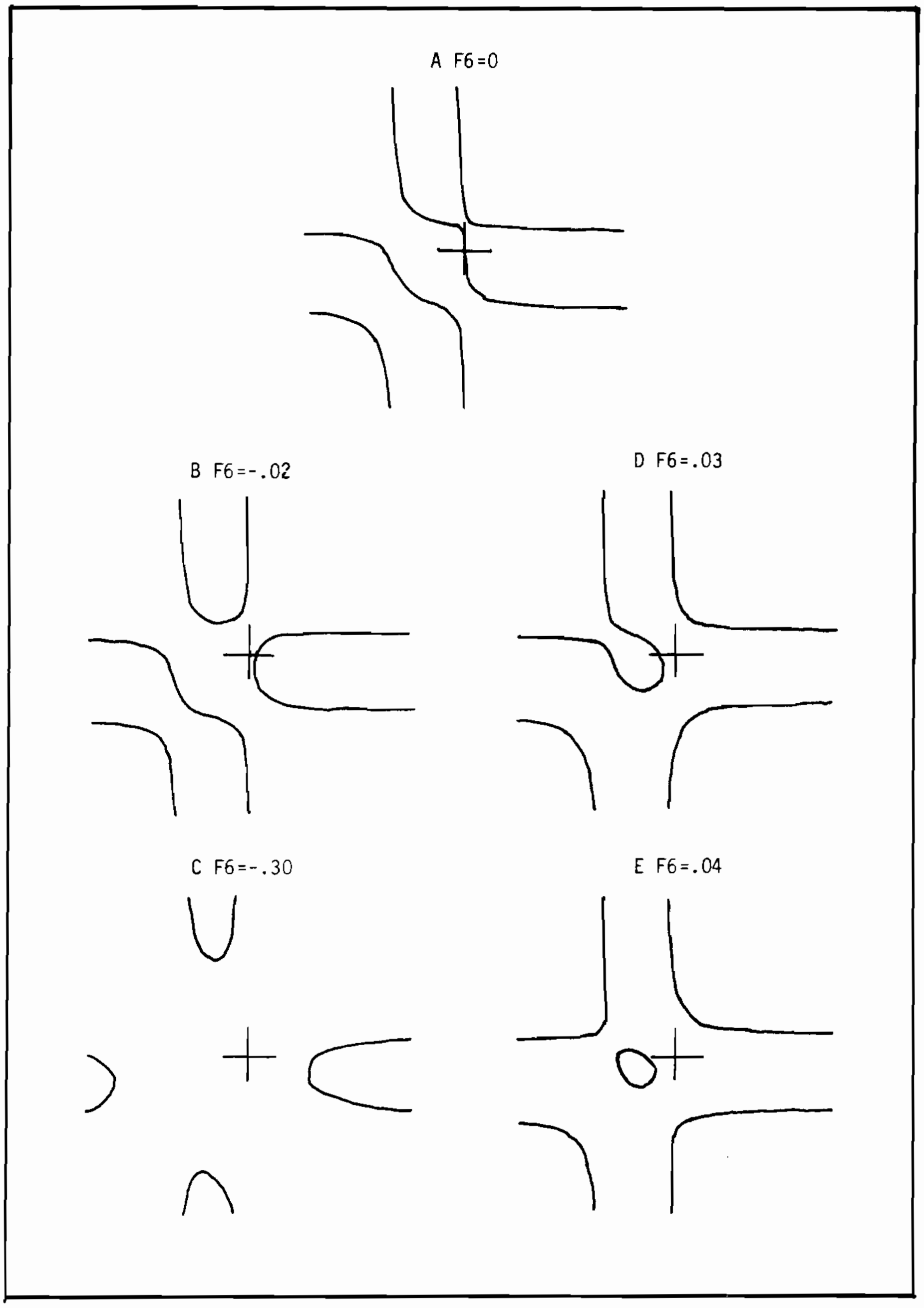

FIGURE 12 Intersections of F6 with the plane z = -.6, at various levels of F6. The graphs show branches 3 4 5 6. Axes are cut-off at ± 3. Vertical axis is x, horizontal y. A cross identifies the point where x = y = 0. Further explanation in the text of section 13.8.

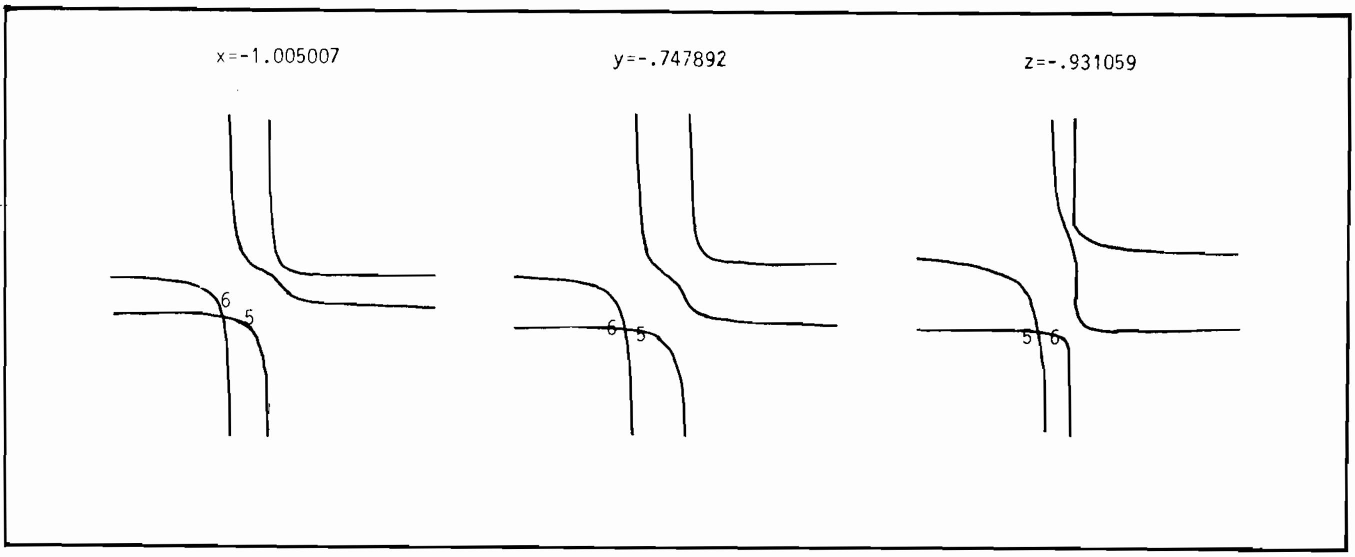

FIGURE 13 Intersections of F6 = 0 at QSP11. In the figures the location of the projections of MB5 and MB6 are roughly indicated by the numbers 5 and 6. In Figures 13 to 19 coordinates are cut off at values ± 3. For x fixed, y is the vertical axis, z horizontal. For y fixed: x vertical, z horizontal. For z fixed: x vertical, y horizontal.

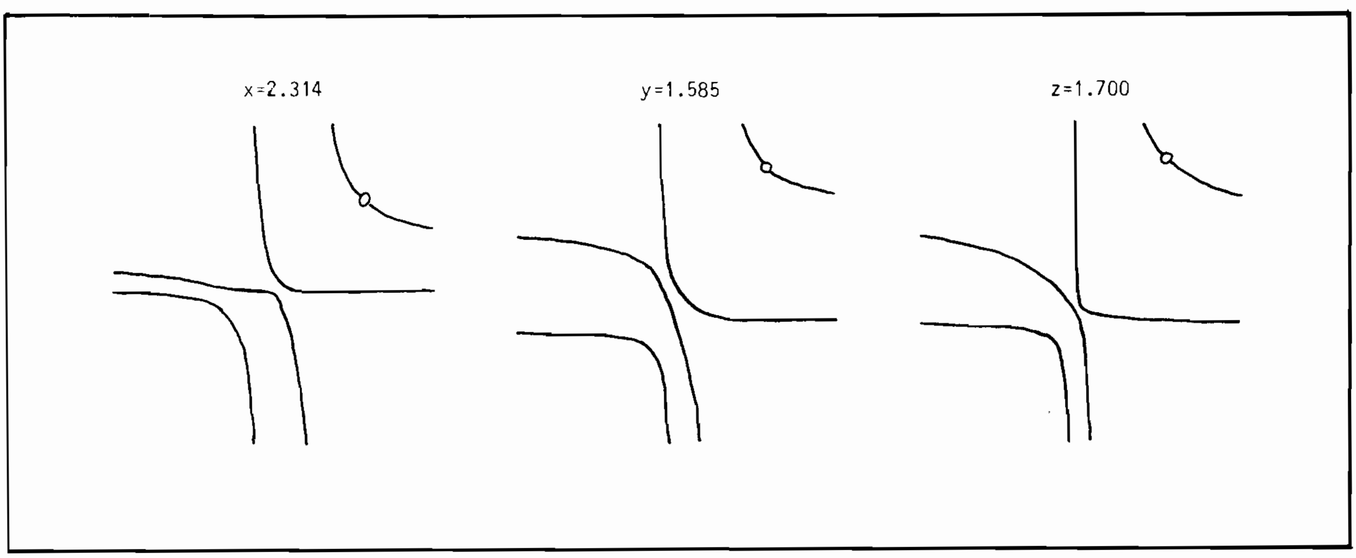

FIGURE 14 Graphs for MB1, with branches 1 2 3 4.

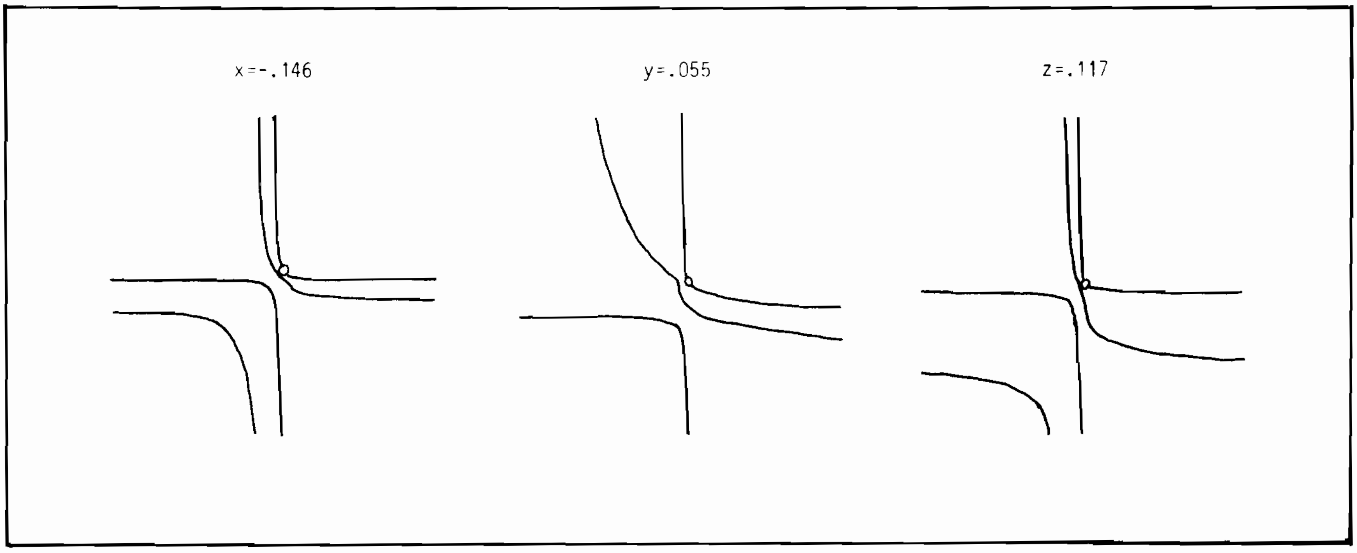

FIGURE 15 Graphs for MB2, with branches 2 3 4 5. In the graph for y, the fifth branch at lower left falls outside the frame of the picture.

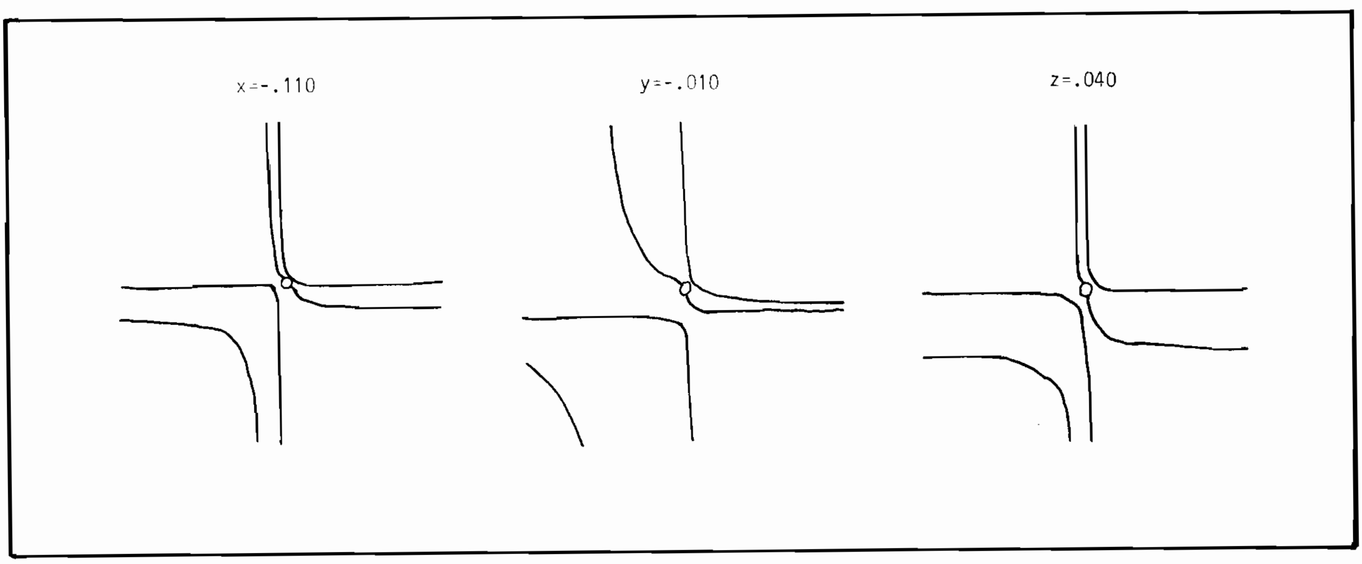

FIGURE 16 Graphs for MB3, showing branches 2 3 4 5.

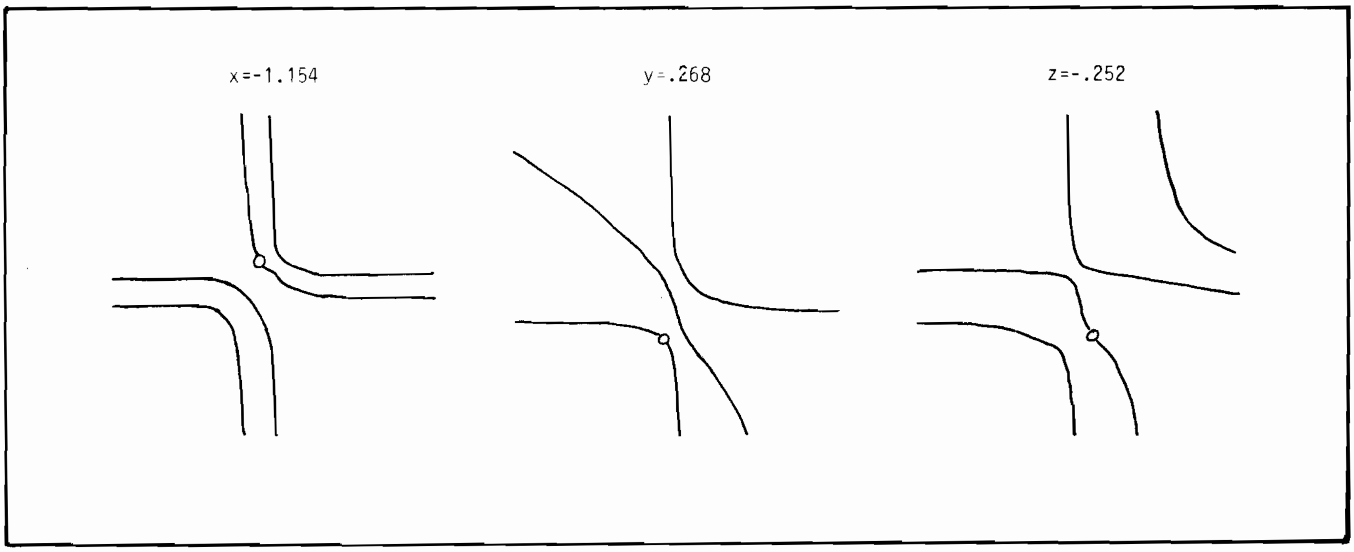

FIGURE 17 Graphs for MB4. Branches 3 4 5 6 in the graph for x, branches 1 2 3 4 in the graph for y (but branch 1 at upper right is outside the frame of the figure), and branches 2 3 4 5 in the graph for z.

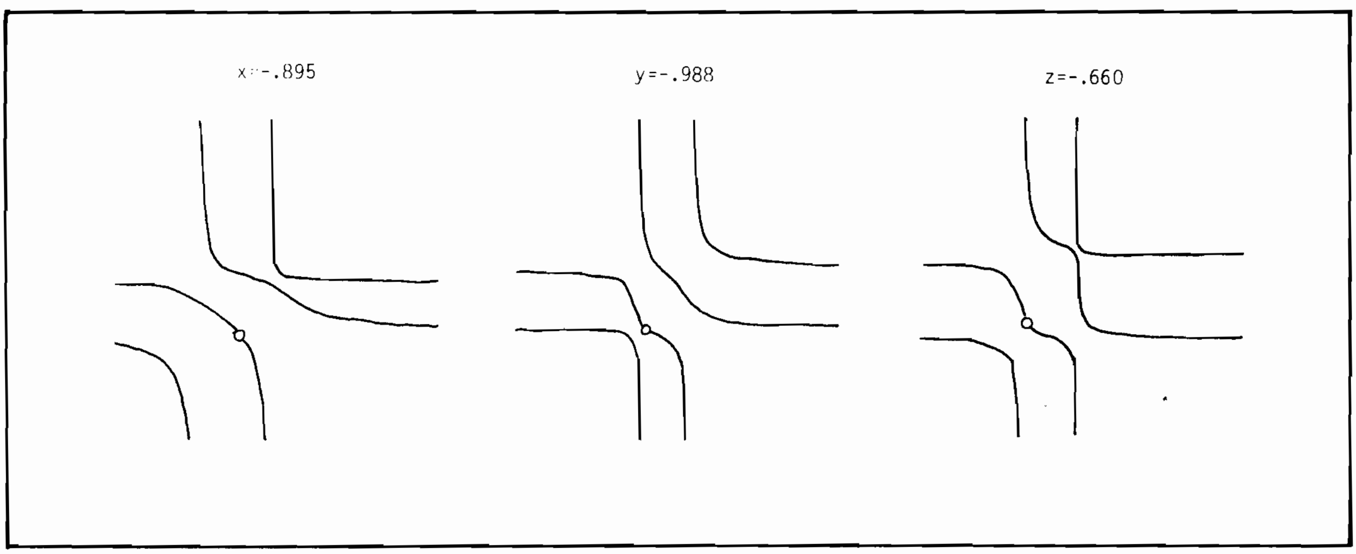

FIGURE 18 Graphs for MB5, showing branches 3 4 5 6.

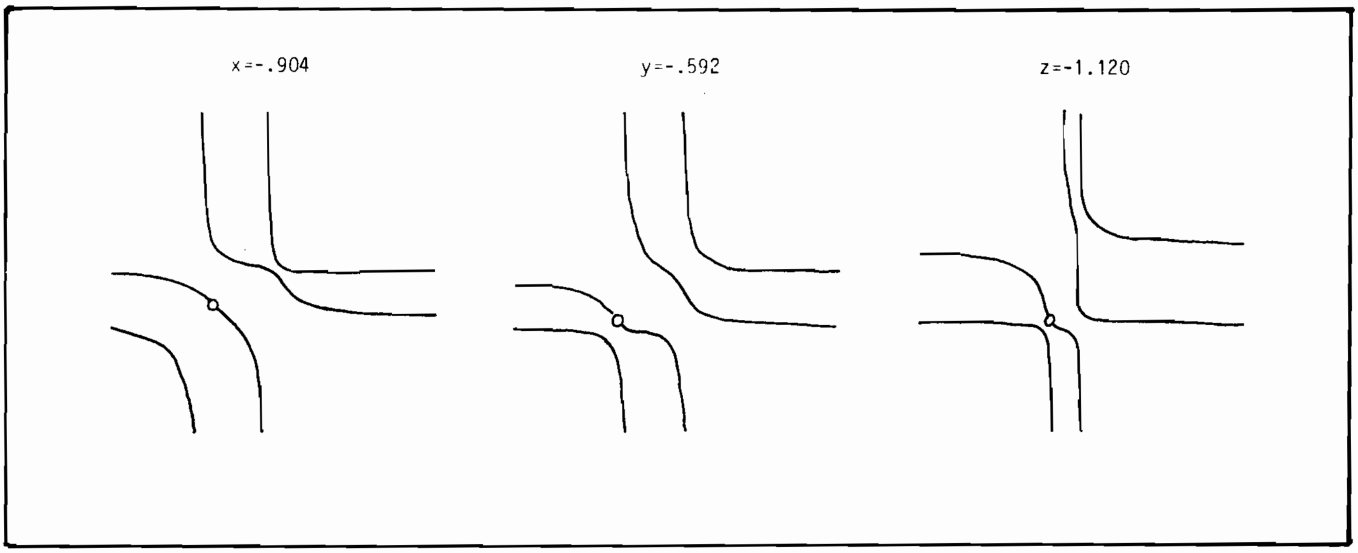

FIGURE 19 Graphs for MB6, showing branches 3 4 5 6.