ANALYSIS OF ASYMMETRY BY A SLIDE VECTOR

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Abstract

The slide vector model is a multidimensional scaling model able to account for asymmetry of a proximity matrix. The asymmetry is modeled by a uniform shift of the object points. No method for fitting the slide vector model seems to be available in the literature. The slide vector model can be viewed as a constrained version of the unfolding model, which suggests an algorithm for fitting the model. A three-way generalization of the slide vector model is proposed. Two examples from market structure analysis are presented.

key words: slide vector model, constrained multidimensional scaling, asymmetry

1. Introduction

Broadly defined, the term multidimensional scaling (MDS) indicates a class of techniques that represent dissimilarities between n objects by distances between points in a possibly low dimensional space. The objects can be anything: cars, personality traits, ethnic categories, and so on. There are various methods for obtaining dissimilarity measures (e.g., Coxon, 1982); some methods of data collection yield *asymmetric* dissimilarity measures. It can be argued that a distance model is not appropriate for the analysis of square asymmetric data tables, because by definition a distance function is *symmetric*. Analyzing an asymmetric matrix by a symmetric model disregards the asymmetry which may carry interesting information. A number of methods has been proposed in the literature to overcome this problem. A review of these methods can be found in, e.g., Zielman (1991).

Examples of asymmetric matrices which might lead to interesting results when subjected to a scaling analysis are: occupational mobility tables, brand switching data, sociometric interaction data, and communication and volume flows. The dissimilarity between object i and j will be denoted by δ_{ij} with (i=1,...,n; j=1,...,n) An object i is said to dominate object j if it is observed that $\delta_{ij} > \delta_{ji}$.

A relatively unknown asymmetric adjustment of the distance model, suggested by Kruskal (1973, personal communication to De Leeuw) will be discussed in this paper. The formula of the model is given in De Leeuw and Heiser (1982) where it is called the *slide vector model*. In Carroll and Wish (1974) the *drift vector model* is mentioned. According to the description in Carroll and Wish (1974) the slide vector model and drift vector model probably refer to the same model; this paper uses the name slide vector model. The slide vector model represents asymmetry by a uniform shift, or a translation, of the difference vector between the points in a multidimensional space.

In this paper an algorithm based on the majorization theory of De Leeuw and Heiser (1980) is presented for fitting the slide vector model. In addition, a three-way variant of Kruskal's model will be proposed.

2. Modelling asymmetry by a slide vector

The slide vector model is a multidimensional scaling model that can account for asymmetry in an observed proximity matrix. The slide vector model represents the asymmetry in the data by a uniform shift of the object points in one direction. The model is:

$$d_{ij}(\mathbf{X};\mathbf{z}) = \sqrt{\Sigma_s(x_{is} - x_{js} + z_s)^2}. \quad (1)$$

The modified distance function is a function of the configuration matrix **X**, with coordinates x_{is} (i=1,...,n; s=1,...,p) and the slide vector **z**, with coordinates z_s. The parameter p denotes the dimensionality of the model. The model contains the simple Euclidean model as a special case if the slide vector is equal to zero. Note that the diagonal elements of the model are non-zero constants, unlike a regular distance model. Furthermore, if coordinates coincide, the model predicts a nonzero symmetric dissimilarity between the objects.

Inserting $x_{js} - z_s = y_{js}$ into (1) yields:

$$d_{ij}(\mathbf{X};\mathbf{Y}) = \sqrt{\Sigma_s\ (x_{is} - y_{js}\ )^2}. \quad (2)$$

Distance formula (2) is used in multidimensional unfolding (MDU). From this equation it follows that the slide vector model is a special case of the unfolding model; i.e., an unfolding model where the configuration matrix **Y** for the columns is a translation of the configuration matrix **X** for the rows. The unfolding model merely assumes that there is a configuration for the rows and a configuration for the columns; in general, these two configurations do not need to be a function of each other. The relations in the data table are described by the distances between the row and the column configuration. A disadvantage of the unfolding model is that this model uses twice the number of parameters as compared to a symmetric MDS method. A possible consequence is that the diagrams obtained from an MDU analysis may be difficult to understand. In the context of analyzing square asymmetric matrices the unfolding method is a column-specific slide vector model: every column point $\mathbf{y}_j$ can always be decomposed as the sum of the row point $\mathbf{x}_j$ and a slide vector $\mathbf{z}_j$.

Suppose there are multiple tables available for analysis. For instance data may have been collected for different subjects, under different conditions or at different points in time. The three-way generalization of the slide vector model with a diagonal transformation matrix $\mathbf{U}_k$ that we wish to propose here is:

$$d_{ij}(\mathbf{X};\mathbf{U}_k;\mathbf{z}) = \sqrt{\Sigma_s\ u_{ks}(x_{is} - x_{js} + z_s)^2}, \tag{3}$$

where u_{ks} is a weight indicating the relative importance of the dimension for the kth data source (k=1,...,m). The configuration matrix **X** is usually called the *common space*. The matrix $\mathbf{U}_k$ can be diagonal; in this case an asymmetric INDSCAL model is obtained. If the matrix is a p by p square matrix an asymmetic extension of the IDIOSCAL model is obtained. The model assumes a set of p dimensions underlying the n objects in the same way as in the simple slide vector model. They are assumed to be common to all sources, hence the name common space. Formula (3) differs from (1) only in the presence of the dimension weights u_{ks}. The dimensions and the transformation-weights can be interpreted analogously to the weights from the INDSCAL model (Carroll and Chang, 1970; Carroll, 1972) for symmetric dissimilarity data.

3. Geometry of the slide vector model

An attractive feature of the slide vector model is that the objects are depicted in a low dimensional space and the slide vector is incorporated as an additional point in this space. In contrast with other models (Gower, 1977; Weeks and Bentler, 1982; Okada, 1988 a,b; Zielman, 1991) the asymmetry parameters are linked to the *dimensions* of the scaling model, not to the objects themselves. The geometry of the slide vector model is illustrated in Figure 1.

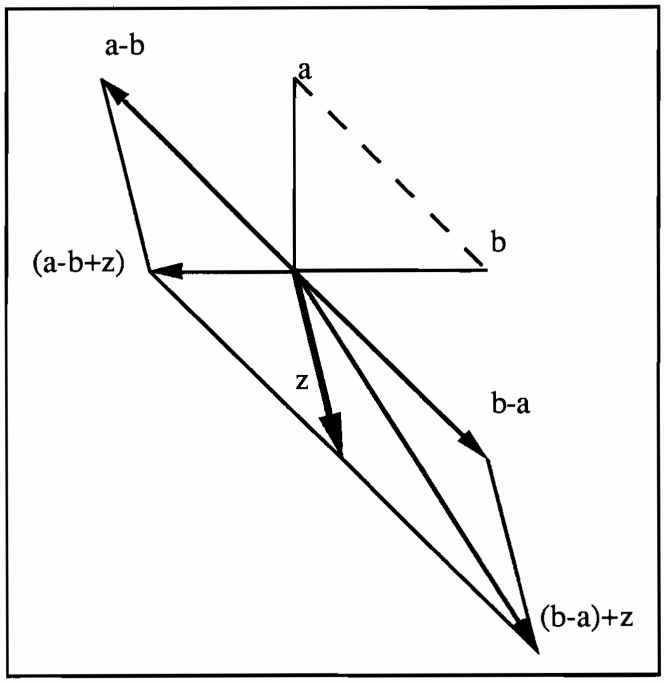

Figure 1 Geometry of the slide vector model

In Figure 1 two objects are depicted as vectors in a two dimensional space; the displacement between the termini of the vectors is drawn by a dashed line. The distance is computed by first subtracting the vectors (a-b, b-a); these difference vectors are of the same length but with opposite sign. By adding the slidevector, indicated by the bold vector z in the Figure, to these difference vectors, and then establishing their length, an asymmetric quasi-distance ((b-a)+z,(a-b)+z) is obtained. From Figure 1 it follows that objects located in a direction similar to the slide vector dominates the other objects. From the MDU interpretation it follows that the column points are translated in a specific direction.

The interpretation of a joint plot of the configuration and the slide vector can be simplified by distinguishing a *preference* part indicating the dominant member of a pair of objects and a *symmetric* part indicating the similarity between these objects. Squaring equation (1) and expanding the square yields:

$$d_{ij}^2(\mathbf{X};\mathbf{z}) = \Sigma_s\,(x_{is} - x_{js})^2 + \Sigma_s\, z_s^2 + 2\Sigma_s\, z_s\,(x_{is} - x_{js}). \tag{5}$$

The first and second terms on the right-hand side of (5) are symmetric; the first term corresponds to the squared Euclidean distance indicating the similarity between the objects. The second term indicates the length of the slide vector, this corresponds to the general amount of asymmetry explained by the model. The third term on the right hand side of (5) is the preference part, it has the property of *skew-symmetry*; this means that $\Sigma_s z_s (x_{is} - x_{js}) = -\Sigma_s z_s (x_{js} - x_{is})$. This skew-symmetric term corresponds to the Vector model of preference data (Tucker, 1960). The dominance of an object with respect to the other objects can be found by the projection of the object on the slide vector. The vector model interpretation is illustrated in Figure 2.

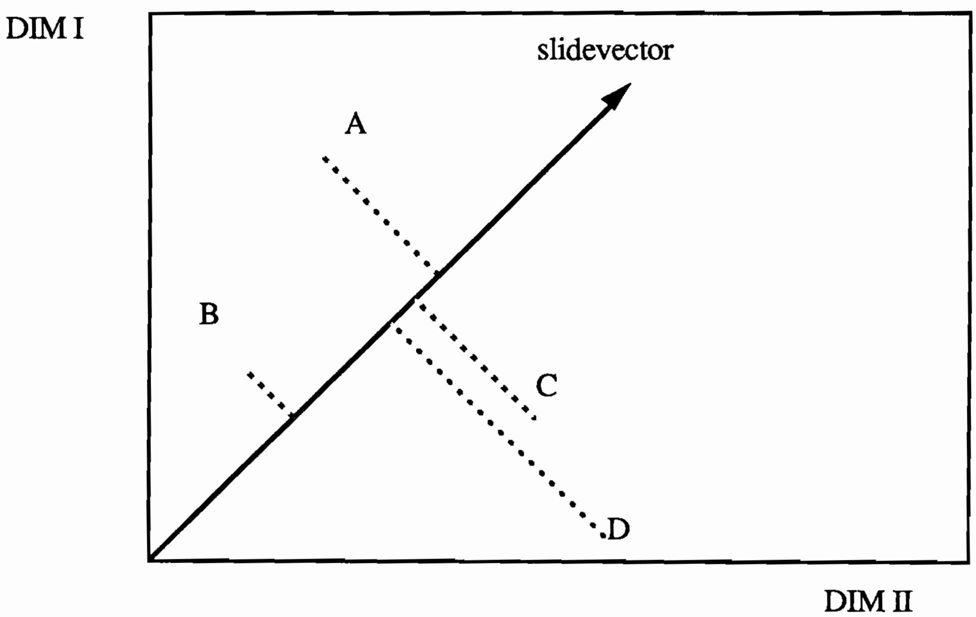

Figure 2 Joint representation of the symmetry and asymmetry

In Figure 2 four objects A,B,C,D are depicted as points in a two dimensional space. The dashed lines in Figure 2 correspond to the projections of the objects on the slide vector. Objects with high projections along the slide vector dominate objects with low projections.

In this example object A dominates the other objects. The distances among the points can be interpreted as the similarity or resemblance of the objects; object C is more similar to object D than to object A. In this small example the quasi-distance, $d_{ij}(\mathbf{X};\mathbf{z})$, from point A to point C is larger than the quasi-distance $d_{ji}(\mathbf{X};\mathbf{z})$, from point C to point A.

The dimensions of the two-way model can always be rotated in such a way that z_s is zero, except for one dimension. This rotational freedom is lost in the three-way model. For the INDSCAL model the asymmetry can be displayed by vectors with coordinates that can be found in the diagonal elements of $\mathbf{U}_k$. The dominating objects can now be found by projecting the objects along the vector for each occasion.

4. A restricted unfolding algorithm

In this section it will be shown how the coordinates and the slide vector of the various models can be obtained by using an unfolding algorithm that can handle restrictions on the configuration. A detailed description of this algorithm for unfolding can be found in Heiser (1987). How well a particular configuration matches the data will be measured by the STRESS badness-of-fit index (Kruskal, 1964 a,b):

$$\sigma(\mathbf{X};\mathbf{Y}) = \Sigma_i\Sigma_j w_{ij}(\delta_{ij} - d_{ij}(\mathbf{X};\mathbf{Y}))^2. \qquad (6)$$

The weights w_{ij} in (6) can be used to code missing data, or to simulate the behavior of other loss functions. In our application the weights will be used to suppress the effects of the diagonal elements of the data matrix. The iterative majorization approach to MDS of De Leeuw and Heiser (1980), leading to the so-called SMACOF algorithm, basically consists of a repetition of two steps: first improve the locations of the points in the multidimensional space by computing the Guttman transform; and second take care that these improved locations satisfy the desired model restrictions by solving a metric projection problem. A third step can be added to this two-step algorithm by introducing optimal scaling of the dissimilarities, but this generality will not be discussed in the sequel.

In the unfolding case the Guttman transform can be split up into a Guttman transform for the row points and a Guttman transform for the column points. First we define the auxiliary matrix **A** with elements:

$$a_{ij} = \frac{w_{ij}\delta_{ij}}{d_{ij}(\mathbf{X};\mathbf{Y})} \qquad \text{if } d_{ij}(\mathbf{X};\mathbf{Y}) > 0,$$

$$a_{ij} = 0 \qquad \text{if } d_{ij}(\mathbf{X};\mathbf{Y}) = 0.$$

The weights are collected in a matrix $\mathbf{W} = \{w_{ij}\}$, and four diagonal matrices are defined as follows:

$$\mathbf{P} = \text{diag}(\mathbf{Ae}),$$

$$\mathbf{Q} = \text{diag}(\mathbf{e}'\mathbf{A}),$$

$$\mathbf{R} = \text{diag}(\mathbf{We}),$$

$$\mathbf{C} = \text{diag}(\mathbf{e}'\mathbf{W}),$$

where **e** denotes the n-vector of ones, and diag ( ) denotes a diagonal matrix having diagonal elements equal to its vector argurment. The preliminary matrices **X*** and **Y*** are computed from previous estimates $\mathbb{X}$ and $\mathbb{Y}$ as follows:

$$\mathbf{X}^* = \mathbf{P}\,\mathbb{X} - \mathbf{A}\mathbb{Y},$$

$$\mathbf{Y}^* = \mathbf{Q}\mathbb{Y} - \mathbf{A}'\mathbb{X}.$$

The unconstrained updates $\mathbf{X}^+$ and $\mathbf{Y}^+$, of the row and column configurations (The Guttman transforms), can be found by solving the system of equations (Heiser, 1987):

$$\mathbf{RX}^+ - \mathbf{WY}^+ = \mathbf{X}^*$$

$$\mathbf{CY}^+ - \mathbf{W'X}^+ = \mathbf{Y}^*$$

The final step in the SMACOF algorithm is: as long as the improvement of STRESS exceeds a predetermined small positive constant, set $\mathbb{X}$ and $\mathbb{Y}$ equal to $\mathbf{X}^+$ and $\mathbf{Y}^+$, respectively, and compute new updates. A proof of convergence can be found in De Leeuw and Heiser (1980) and De Leeuw (1988). To incorporate the restrictions imposed by the slide vector model a metric projection problem (De Leeuw and Heiser, 1980) has to be solved. This can be done by introducing the auxiliary matrix $\mathbf{E}$, of order $2n$ by $n+1$, having the structure:

$$\mathbf{E} = \begin{pmatrix} \mathbf{I} & \mathbf{e} \\ \mathbf{I} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{e}$ is a vector of ones. Secondly we define the matrix $\mathbf{S}$ of order $2n$ by p in which the matrices $\mathbf{X}^+$ and $\mathbf{Y}^+$ are stacked on top of each other. The matrix $\mathbf{E}$ can be regarded as a design matrix where the first n columns code the equality constraints on the coordinates for the row and column points. The weights are recorded in a new matrix $\mathbf{V}$ which is build up from the old weight matrices; the $\mathbf{V}$ matrix is partioned as:

$$\mathbf{V} = \begin{pmatrix} \mathbf{R} & -\mathbf{W} \\ -\mathbf{W'} & \mathbf{C} \end{pmatrix}.$$

The metric projection problem can now be formulated as follows: estimate the matrix $\mathbf{B}$ that minimizes:

$$L(\mathbf{B}) = \mathrm{tr}\,(\mathbf{S} - \mathbf{EB})'\mathbf{V}(\mathbf{S} - \mathbf{EB}). \qquad (7)$$

The solution of the metric projection problem (7) is:

$$\mathbf{B} = (\mathbf{E}'\mathbf{VE})^{-}\mathbf{E}'\mathbf{V}'\mathbf{S}.$$

The matrix $(\mathbf{E}'\mathbf{VE})^{-}$ denotes the generalized inverse of the matrix $(\mathbf{E}'\mathbf{VE})$. This inverse can be computed as $\{(\mathbf{E}'\mathbf{VE})+\alpha(\alpha'\alpha)^{-1}\alpha'\}^{-1} - \alpha(\alpha'\alpha)^{-1}\alpha'$ where α spans the null-space of the matrix $(\mathbf{E}'\mathbf{VE})$; this vector α is an $n+1$ vector with unities and in the last position a zero. The remaining part of the algorithm now is to compute **EB** and set **X** equal to the first n rows of this matrix and set **Y** equal to the last n rows of this matrix. With these constrained **X** and **Y** matrices we can compute new Guttman transforms. The slide vector is given as the last row of the matrix **B**.

Next it will be shown for the three-way slide vector model how the INDSCAL and IDIOSCAL model can be estimated with the SMACOF algorithm by solving another metric projection problem. The INDSCAL model allows a differential weighting of the dimensions by different data sources. The IDIOSCAL model allows a differential rotation of the dimensions as well. The model is extended with an additional matrix $\mathbf{U}_k$ incorporating the various restrictions required by the individual differences model. The metric projection problem can now be formulated as:

$$L(\mathbf{B};\mathbf{U}_k) = \frac{1}{m}\mathrm{tr}\Sigma_k\,(\mathbf{S}_k - \mathbf{EBU}_k)'\mathbf{V}_k(\mathbf{S}_k - \mathbf{EBU}_k), \qquad (8)$$

where $\mathbf{S}_k$ contains the stacked Guttman transforms for the kth data source, and $\mathbf{V}_k$ denotes the weight matrix for the kth datasource. The matrix product **EB** is the common space. This common space satisfies the slide vector restrictions. The matrix $\mathbf{U}_k$ contains the linear transformation of the common space for source k. The matrix $\mathbf{U}_k$ can be constrained to be a diagonal matrix in which case we obtain an INDSCAL representation. If this matrix is free, an IDIOSCAL representation is obtained (see Heiser and Stoop, 1986). Loss function (8) can be minimized by Alternating Least Squares (ALS); for fixed $\mathbf{B}$ the solution of $\mathbf{U}_k$ is, assuming the inverse exists:

$$\mathbf{U}_k = (\mathbf{B'E'V}_k\mathbf{EB})^{-1}\mathbf{B'E'V}_k\mathbf{S}_k$$

For fixed $\mathbf{U}_k$ the matrix $\mathbf{B}$ can be found by solving the system of equations:

$$\Sigma_k\{\mathbf{U}_k \mathbf{U'}_k \otimes \mathbf{E'V}_k\mathbf{E}\} \text{ vec }(\mathbf{B}) = \text{vec }(\Sigma_k \mathbf{E'V}_k\mathbf{SU}_k), \quad (9)$$

where the symbol ⊗ denotes the Kronecker product of matrices; for instance $\mathbf{A} \otimes \mathbf{B} = [a_{ij}\mathbf{B}]$, and vec ( ) denotes the operator that transforms a matrix into a vector by stacking the columns of a matrix one underneath the other. If the individual differences are modeled by weighting the dimensions, the estimation problem of the common space reduces to p smaller subproblems; because of the blockwise structure of the matrix on the left hand side of (9) the common space coordinates can be estimated dimension by dimension.

5. Two Examples

In this section we will discuss two applications from market structure analysis. The aim of market structure analysis is to partition brands into sub-markets with close competition. The data of the first example are taken from a study by Harshman e.a. (1982), and deal with the phenomenon of brand switching between 16 car segments. The entries in the table indicate the number of people who bought brand i in the first period and currently buy brand j in the second period. Brand switching data are likely to be nonsymmetric; usually a different proportion of brand A users switch to brand B than vice versa within a given time period. A detailed description of the data can be found in Harshman e.a. (1982). The brands are positioned in a multidimensional space where brands located close together compete more with each other than with brands far apart. The slide vector represents the direction in the space where the most popular brands, in the sense of more "switched to" than "switched from" can be found.

The raw brand switching matrix is not appropriate as input for the scaling program because the data have to be converted from similarities to dissimilarities. In most applications it is desirable to adjust for large differences in market share. These two steps can be performed by applying the gravity model (Tobler and Wineburg, 1971). This amounts to first dividing the raw frequencies n_{ij} by their row and column sums; and, secondly the standardized frequencies are inverted. These inverted numbers should yield *squared* distances, according to the gravity model, so as a last step the square root of the quantities is taken. The diagonal of the matrix is non-zero; this may complicate our analysis because these numbers will influence the length of the slide vector. This effect is eliminated from the analysis by giving these diagonal elements zero weights.

The transformed car switching matrix was analyzed in two dimensions. According to the Gower (1977) decomposition the skew-symmetric variation in the data was 4 percent. The coordinates for two analyses are reported in Table 1; the coordinates from the unconstrained analysis are graphically displayed in Figure 3. The first three letters of the labels indicated the size of the cars: sub = subcompact; sma = small; com = compact; mid = midsize; std = standard; lux = luxury. The last letter indicates a distinguishing feature within a size category: c = captive import; d = domestic; i = import; s = specialty; m = medium price; l = low price.

Table 1 Configuration matrices car switching data

	unconstrained		constrained	
Segment	DimI	DimII	DimI	DimII
SUBD	-0.125	-0.135	0.134	0.138
SUBC	-0.031	-0.101	-0.021	0.120
SUBI	0.058	-0.125	-0.021	0.120
SMAD	0.012	-0.022	0.067	0.042
SMAC	-0.034	-0.156	-0.088	0.024
SMAI	0.162	-0.065	-0.088	0.024
COML	-0.212	-0.067	0.158	0.181
COMM	-0.140	-0.045	0.158	0.181
COMI	0.090	-0.082	0.003	0.163
MIDD	-0.103	0.039	0.144	-0.035
MIDI	0.125	0.005	-0.011	-0.053
MIDS	0.011	0.123	0.144	-0.035
STDL	-0.192	0.134	0.301	-0.012
STDM	-0.033	0.216	0.301	-0.012
LUXD	0.178	0.221	0.125	-0.204
LUXI	0.235	0.059	-0.030	-0.223
slidevector	0.015	-0.023	-0.042	0.007

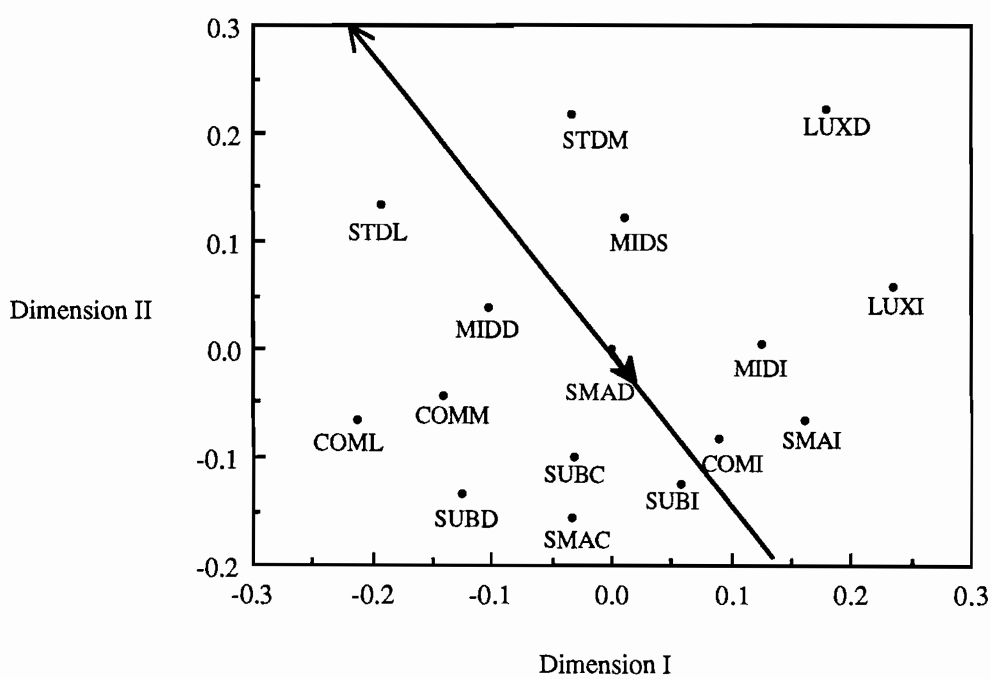

Figure 3 Two-dimensional solution car switching data

The first dimension corresponds to an import-domestic distinction and the second dimension discriminates the small cars from the large cars. The fit of the two-dimensional model is quite good (STRESS = 0.066). According to the Gower (1977) decomposition of the residual matrix the STRESS due to symmetry was 0.035; the STRESS due to asymmetry was 0.031. From Figure 3 we see that the imported cars compete more with each other than with the domestic cars. There is also strong competition among small cars.

The slide vector is depicted as an arrow with a bold tail. The long vector with the plain tail is the "switch from" vector. This arrow is drawn larger to facilitate the interpretation. For instance there are more *switches from* Luxury Domestic to Luxury Import than the other way round. The trend is to buy a luxury imported car instead of a luxury domestic car. There are more switches from the Standard and midsize cars to the small and subcompact cars than the other way round; the trend is that the small cars are winning in the market. These results are in close agreement with the results found by Harshman e.a. (1982) who analyzed these data with the DEDICOM model.

As a second step the data will be re-analyzed by a slightly more constrained version of the slide vector model. For a discussion on constrained MDS, see Heiser and Meulman (1983 a,b). The coordinates of the objects in the space are now required to be a linear combination of external categorical variables, or product related attributes. Within the SMACOF theory this is a relatively simple task; in the first n columns of the matrix **E** are replaced by vectors with unities and zero's. A one indicates the presence of the attribute and a zero indicates the absence of the attribute; see Meulman and Heiser (1984) for details. In the car switching data the external variables are the size of the cars, with categories subcompact, small, compact, midsize, standard and luxury. The second variable is import, with two categories import and domestic. Captive imports are treated as imported cars. The variables are coded as an ANOVA-type design matrix.

The STRESS of the two dimensional constrained solution is 0.11; this is only slightly larger than the unconstrained solution. The STRESS due to symmetry was 0.079; the STRESS due to asymmetry was 0.031. The constrained solution is displayed in Figure 4.

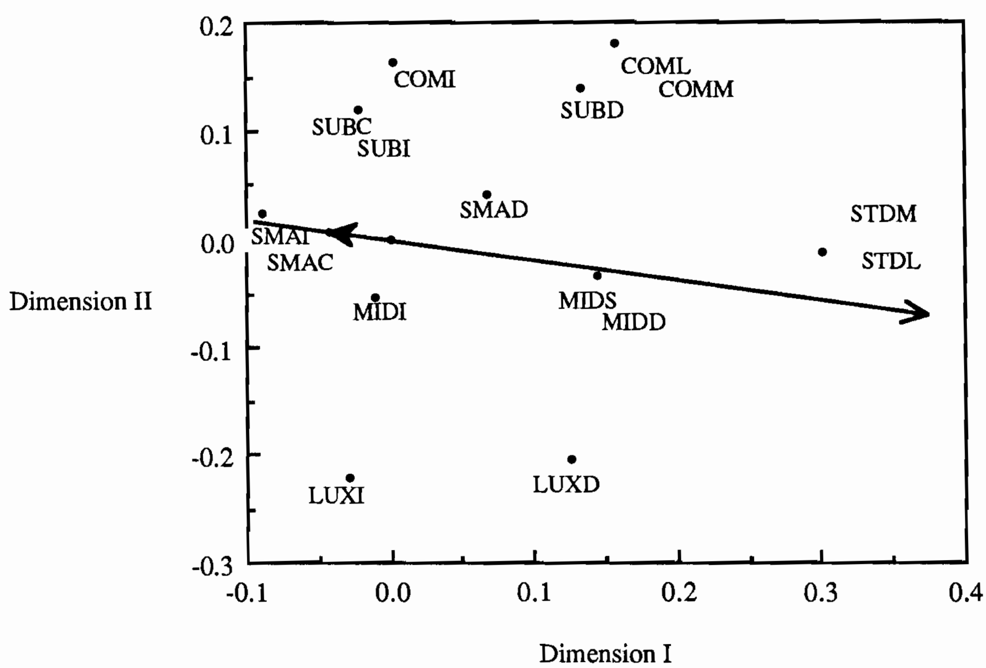

Figure 4 Constrained solution of the car switching data

The horizontal dimension is related to the size of the cars, the vertical dimension can be interpreted as a luxury dimension. The imported cars are located in the left quadrants of the space. The trend is to buy small cars, in 1979 within a class of cars of the same size there were more switches from domestic cars to imported cars than the other way round.

The second example is the analysis of a soft drinks brand switching matrix (Bass, Pessemier, and Lehmann, (1972). The soft drinks have some distinctive features; they differ in flavor and in calories. The data were again preprocessed according to the gravity model. The brand names, coordinates and the design matrix can be found in table 2.

Table 2 Brands, Design and Coordinates soft drink data

	Design		Coordinates	
Brand name	Cola	Diet	Dim I	Dim II
Coke	yes	no	0.039	0.119
7-Up	no	no	0.000	0.000
Tab	yes	yes	-0.015	0.109
Like	no	yes	-0.054	-0.009
Pepsi	yes	no	0.039	0.119
Sprite	no	no	0.000	0.000
Diet Pepsi	yes	yes	-0.015	0.109
Fresca	no	yes	-0.054	-0.009
Slide vector			-0.329	0.103

The horizontal dimension corresponds with a diet/non-diet dimension; the vertical dimension is a cola /non-cola dimension. From Figure 5 we can conclude that there occurs more switching between diet and non diet drinks than between different flavors. There is a trend toward the diet drinks, in the sense that there are more switches from non-diet drinks to diet drinks than the reverse. A smaller trend can be found towards the cola drinks: the cola's become more popular. The fit of the two-dimensional restricted model was satisfactory (STRESS = 0.092). According to the Gower (1977) decomposition of the matrix with residuals the STRESS due to symmetry was 0.071; the STRESS due to asymmetry was 0.011. The STRESS of the unconstrained solution was 0.091.

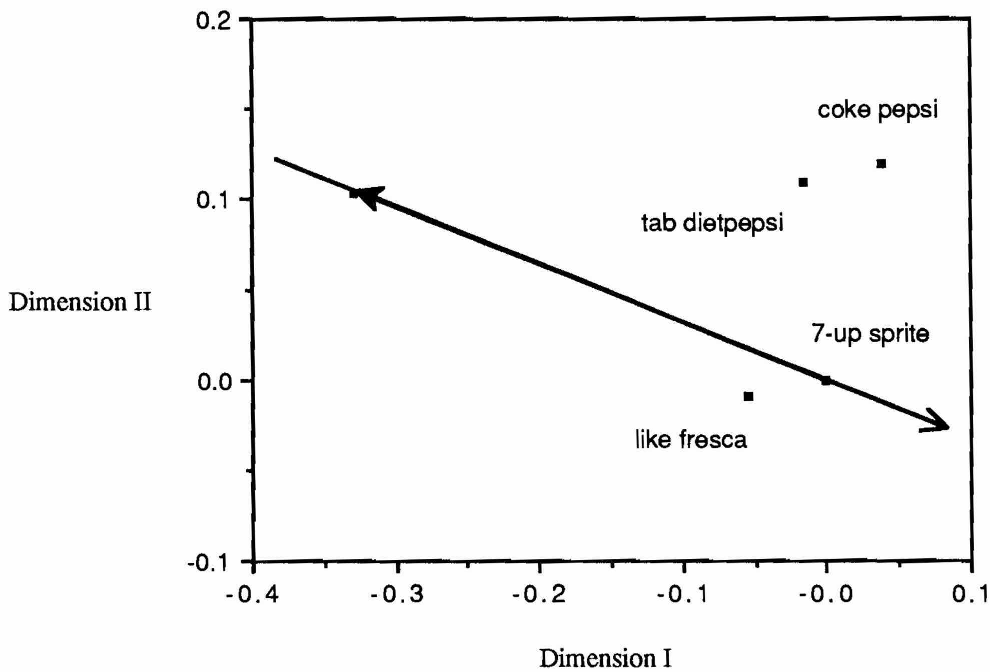

Figure 5 Constrained solution soft drink data

Figure 5 should be interpreted with care, because the distance between Coke and Pepsi appears zero but is actually equal to the size of the slide vector. The direction of the slide vector is determined by the trend in switching behavior, the size is determined by this trend but also by the number of symmetric switches between brands with equality constraints. In figure 6 the MDU interpretation of the model is illustrated.

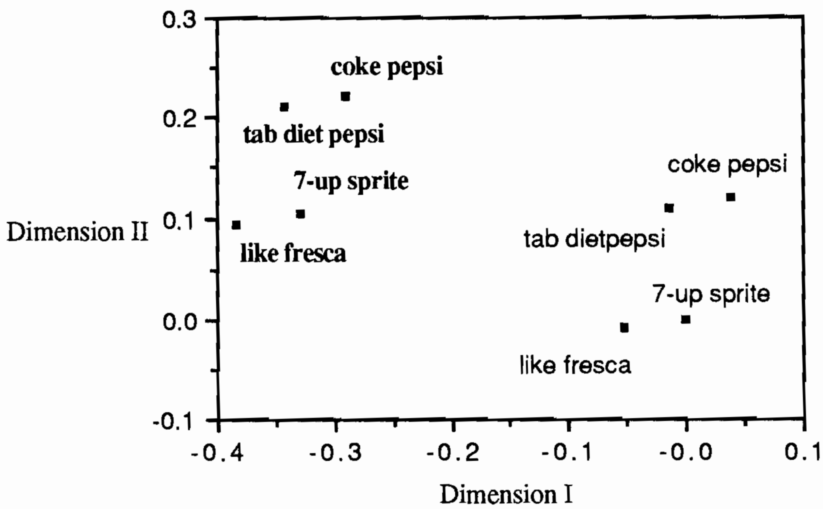

Figure 6 Unfolding representation of the soft drink data

In Figure 6 the plain labels indicate the row points and the **bold** labels indicate the column points. The figure shows the translation of the column points by the slide vector. Because there are just a small number of points the unfolding diagram might be easier to interpret.

6. Discussion

The perspective of constrained unfolding allows us to think of a generalization of the slide vector model, in which there are multiple slide vectors and some points share the same slide vector. The last column of **E** codes the slide vector. If multiple slide vectors would be desired, the matrix **E** has to be augmented with additional columns. This model will be called the *multiple slide vector model.* The model can be written as:

$$d_{ij}(\mathbf{X};\mathbf{z}) = \sqrt{\sum_s (x_{is} - x_{js} + z_{ks})^2}.$$

Where k denotes a set of objects sharing the same slide vector. These sets of objects can be defined by the user, or they can be estimated from the data. Examples of what these sets or classes might be are: a distinction between diet and non diet soft drinks, import and domestic cars in the analysis of brand switching data, or a distinction between "hard" and "soft" psychology journals in the analysis of scientometric transaction matrices. If these sets are defined by the users no new complications arise; if they have to be estimated from the data we have to solve a combinatorial problem.

Another generalization of the slide vector would be the *rowweighted slide vector model*, which can be written as:

$$d_{ij}(\mathbf{X};\mathbf{z};\mathbf{a}) = \sqrt{\sum_s (x_{is} - x_{js} + z_s\, a_j)^2}.$$

Here a_j is a weight indicating the importance of the slide vector for modeling the flow from object i to object j. A reason for using this general model would be to model the diagonal entries, rather than ignoring them as we have done here. Within the SMACOF theory the estimation of such models does not lead to new complications.

A related model is the Jet-stream model proposed by Gower (1977). The Jet-stream model also represents the asymmetry in the data by a direction in the space. The model differs from the slide vector model in the sense that it divides the distance by an asymmetric term instead of adding a term to the distance

Another application of the slide vector could be to paired comparison data from a choice experiment, since we have to deal with asymmetric frequencies here as well. The model then resembles the wandering vector model of Carroll (1981) for paired comparison data, although our model is additive instead of multiplicative. The wandering vector model is designed for skew symmetric data, in an application of the slidevector model an additive constant shoud be added to these data.

Approximate slide vector representations could, of course, be obtained by using an unfolding program without restrictions; however, the present methodology is fruitful because the structure, if present, is sometimes difficult to recognize for the novice user of unfolding. Moreover, the structure might be hidden by random fluctuations in the data.

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