The null-space of a partitioned matrix

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1: Introduction

Suppose

$$M = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \tag{1}$$

where A is $n \times m$, B is $n \times p$, C is $q \times m$, and D is $q \times p$ (all matrices in this paper are real). The null-space of M, written as null(M), is the set of all vectors z in $\mathbf{R}^{m+p}$ such that $Mz = 0$. The nullity of M, written as $\eta(M)$, is the dimensionality of the null-space. The rank of M, written as $\rho(M)$, is defined as $\rho(M) = (m + p) - \eta(M)$.

If A is square and nonsingular, then the following results are well known (Guttman, 1946, compare also the review by Ouellette, 1978):

$$\text{null}(M) = \{\binom{x}{y} \mid x \in \mathbf{R}^m \ \&\ y \in \mathbf{R}^p \ \&\ y \in \text{null}(D - CA^{-1}B) \ \&\ x = -A^{-1}By\}, \tag{2}$$

$$\eta(M) = \eta(D - CA^{-1}B), \tag{3}$$

$$\rho(M) = (m + p) - \eta(M) = \rho(A) + \rho(D - CA^{-1}B). \tag{4}$$

In this paper we generalize these results to the case where A is not necessarily square, and not necessarily of full column rank. Our techniques are closely related to those of a companion paper on partitioned determinants (De Leeuw, 1981).

2: First reduction

We must solve the system

$$Ax + By = 0, \tag{5a}$$

$$Cx + Dy = 0, \tag{5b}$$

with $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$. Suppose $\rho(A) = r$. Then the singular value decomposition of A is $A = K_1\Omega L_1'$, with K_1 an $n \times r$ columnwise orthonormal matrix, L_1 an $m \times r$ columnwise orthonormal matrix, and with Ω diagonal and positive definite of order r. We also define an $n \times (n - r)$ columnwise orthonormal matrix K_0 and an $m \times (m - r)$ columnwise orthonormal matrix L_0, which satisfy $K_0'K_1 = 0$ and $L_0'L_1 = 0$.

Write x in the form $x = L_1u + L_0v$, and substitute in (5). Then

$$K_1\Omega u + By = 0, \tag{6a}$$

$$CL_1u + CL_0v + Dy = 0. \tag{6b}$$

Premultiplying (6a) with K_1' gives

$$\Omega u + K_1'By = 0, \tag{7}$$

or

$$u = -\Omega^{-1}K_1'By. \tag{8}$$

Premultiplying (6a) with K_0' gives

$$K_0'By = 0. \tag{9}$$

Substituting (8) into (6b) gives

$$(D - CA^+B)y + CL_0v = 0, \tag{10}$$

where A^+ is the Moore-Penrose inverse of A. We still have to solve (9) and (10) for $y \in \mathbb{R}^p$ and $v \in \mathbb{R}^{m-r}$. We can then make the substitution

$$x = L_0v - A^+By. \tag{11}$$

Observe that we have not excluded the special cases $r = 0$ or $r = n$ or $r = m$. If $r = 0$ we can take $L_0 = I$ and $K_0 = I$, moreover $A^+ = 0$ in this case. If $0 < r = m < n$, then L_0 has zero columns and the terms with L_0 in (10) and (11) drops out. If $0 < r = n < m$, then K_0 has zero columns, and equation (9) drops out. If $0 < r = n = m$, then both K_0 and L_0 disappear, and we are back in the classical situation discussed in our first section.

3: Second reduction

We now study the $(n - r) \times p$ matrix $K_0'B$ and the $q \times (m - r)$ matrix CL_0. Suppose $\rho(K_0'B) = s$ and $\rho(CL_0) = t$. We use the singular value decompositions $K_0'B = P_1\Psi Q_1'$ and $CL_0 = R_1\Phi S_1'$, and the orthogonal complements P_0, Q_0, R_0, S_0 defined in the same way as in the previous section. Again some of these matrices may have zero columns.

Write y as $y = Q_1e + Q_0f$ and v as $v = S_1g + S_0h$. Then (9) gives $e = 0$ or

$y = Q_0 f$. Substitution in (10) gives

$$(D - CA^+B)Q_0 f + R_1 \Phi g = 0, \tag{12}$$

which gives

$$g = - \Phi^{-1} R_1'(D - CA^+B)Q_0 f, \tag{13a}$$

$$R_0'(D - CA^+B)Q_0 f = 0. \tag{13b}$$

We can also write (13b) as $f = N_0 w$, with N_0 an orthonormal basis for the null space of $R_0'(D - CA^+B)Q_0$.

Thus

$$y = Q_0 N_0 w, \tag{14a}$$

$$x = L_0 v - A^+By = L_0 S_0 h - L_0(CL_0)^+(D - CA^+B)Q_0 N_0 w - A^+BQ_0 N_0 w. \tag{14b}$$

This can be collected in a single expression

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} L_0 S_0 & -L_0(CL_0)^+(D - CA^+B)Q_0 N_0 - A^+BQ_0 N_0 \\ 0 & Q_0 N_0 \end{vmatrix} \begin{vmatrix} h \\ w \end{vmatrix} . \tag{15}$$

The matrix on the right is nonsingular, and thus

$$\eta(M) = \eta(CL_0) + \eta(R_0'(D - CA^+B)Q_0). \tag{16}$$

This can also be written as

$$\eta(M) = ((m - r) - t) + ((p - s) - \rho(R_0'(D - CA^+B)Q_0)), \tag{17}$$

or

$$\rho(M) = r + t + s + \rho(R_0'(D - CA^+B)Q_0). \tag{18}$$

Now (15) generalizes (2), (16) generalizes (3), and (18) generalizes (4), which means that our job is done. Results equivalent with (18) have been proved earlier by Meyer (1973) and Marsaglia and Styan (1974), compare also Ouelette (1978, p 48-53). Our results have been proved by using singular value decompositions, but observe that the final formulas only involve matrices with zero subscript, i.e. orthonormal bases for the null-spaces of certain matrices. Clearly the orthonormality is not needed in actual computations, it suffices to use arbitrary bases for those null-spaces.

References

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